

Theoretical Astrophysics

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Introduction

The purpose of this course is to give third year Bachelor students or first year Master students the foundations of theoretical astrophysics, with a broad overview of astrophysical fluid dynamics, radiative transfer, self-gravitating systems (in or out of equilibrium) and collisionless systems. Throughout the course, we will use a methodology that is based on the kinetic theory of an ensemble of particles, these particles being either atoms and molecules, photons, or even stars. This common framework will be very useful for future PhD students in astronomy and astrophysics, aiming at studying stellar structure, star formation and the interstellar medium, galaxy formation and cosmology.

It is strongly required that students attending this course already have a basic knowledge of quantum mechanics and electrodynamics, as well as some basics in fluid mechanics and thermodynamics. Some familiarity with tensor calculus is also required. During this course, we will try to derive self-consistently all the equations, with a minimal reference to a priori knowledge in physics. The level of mathematics required is typical of a bachelor in Mathematics or Physics, not more than that. This course has been designed based on other popular text books, of which we give the list below.

- The Physics of Astrophysics. Volume I: Radiation, Frank H. Shu
- The Physics of Astrophysics. Volume II: Gas Dynamics, Frank H. Shu
- Physics of shock waves and high-temperature hydrodynamic phenomena, Ya. Zel'dovich and Yu. Raiser
- Foundations of Radiation Hydrodynamics, Dimitri Mihalas and Barbara Weibel Mihalas
- Radiative Processes in Astrophysics, George Rybicki and Alan Lightman
- Galactic Dynamics, Second Edition, James Binney and Scott Tremaine

Reading these books is not required, but could be useful in case one wants to dive deeper into one of the topics presented here. This course will be divided into 3 main parts, each associated with one particular type of fluid or particles:

1. astrophysics of gases and fluids (chapters 1 and 2)
2. astrophysics of radiation (chapter 3)
3. astrophysics of collisionless fluids (chapter 4)

The first part, sometimes referred to as *astrophysical fluid dynamics*, will cover the basics of the microscopic description of a gas. We will describe the *kinetic theory* of gases, trying to present the main equation, namely the Boltzmann equation, and many of its properties. We will spend some time computing the moments of the Boltzmann equation (the very meaning of moment will become clear in the relevant sections), which will serve as the foundations of

the Euler equations for fluids. A very important aspect of the Euler equations is their validity range, and the notion of Local Thermodynamical Equilibrium (LTE). We will then derive self-consistently a first-order approximation of the full Boltzmann equation, leading to a consistent derivation of fluid viscosity and heat flux. We will then focus on astrophysical applications, trying to derive analytically *equilibrium solutions* of the Euler equations in presence of gravity or rotation, leading to important astrophysical results to describe the internal structure of stars or accretion discs. One guiding principle of this course will be to analyse the stability of these equilibrium solutions, checking indeed that these solutions do exist in nature. This will lead us to the description of *waves and shocks* in fluids, as well as famous *hydrodynamical instabilities*.

The second part will be the realm of *radiative processes*. Here again, we will start at the microscopic level, introducing the radiative transfer equation, and a few analytical solutions describing many radiation flows. One key application will be to describe the interaction between radiation and matter through *absorption and emission processes*. This will lead us to a more macroscopic description of radiation, here again using the moments of the radiation field. We will use some very basic concepts of quantum mechanics and electrodynamics to fully account for all main astrophysical radiation fields.

The third and final part of the course will be dedicated to *collisionless fluids*. As opposed to atoms and photons that interact a lot with one another, the particles of collisionless fluids never suffer any binary collision. This is an extreme case, which is relevant for stars in massive galaxies, or dark matter particles, possibly made of a new and exotic type of particle. Here again, we will follow the same methodology, starting at the microscopic level, then deriving the properties of the distribution of these collisionless particles, and ultimately defining the moments of their distribution function.

As you will discover, a very similar methodology will be used throughout the course, where radiation physics echoes with rarefied gas or even stellar dynamics. Many useful tools will be introduced and used in detailed applications for astrophysics. Our objective would be ideally that, after this course, you will be able to sustain a conversation with a professional astronomer on a black board, about stars or galaxies, fully equipped with equations and concepts for the future theoretical astrophysicists you will surely become.

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Chapter 1

Kinetic theory in a nutshell

1.1 Introduction

In this chapter, we will focus on a microscopic description of astrophysical gases, using a very powerful methodology called kinetic theory. In most, if not all cases, we will consider atoms and molecules to be point-like particles moving in space and interacting through collisions. Describing these collisions as accurately as possible will be our main task. This will lead us to one of the most famous equations in physics, namely the Boltzmann equation, which describes the dynamics of the particle distribution function. The concept of distribution functions has been introduced in your previous mathematics lectures, and is absolutely central to this course. A particular distribution function is the Maxwell-Boltzmann equilibrium solution, which is the valid solution when the microscopic state of the gas is at the so-called local thermodynamical equilibrium (LTE). This particular but also very common limiting case will lead us to the self-consistent derivation of the Euler equations, the mathematical description of fluid dynamics at the macroscopic level. This transition from microscopic to macroscopic is another key aspect of this course, which is performed using the velocity moments. But under certain conditions, the microscopic state of the gas cannot be considered to be at LTE. We will show that one can still describe the fluid using the Euler equations, but adding as first order corrections viscosity and thermal conduction.

1.2 Particle distribution function

1.2.1 Definition

The number of particles in phase space $(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^6$ is described by the particle distribution function $f(\mathbf{x}, \mathbf{u}, t)$. With $d^3x d^3u$ a volume element in phase space, $dN = f(\mathbf{x}, \mathbf{u}, t) d^3x d^3u$ gives the number of particles with positions contained in the infinitesimal volume $[x, x + dx] \times [y, y + dy] \times [z, z + dz]$ and velocities contained in the infinitesimal volume in $[u_x, u_x + du_x] \times [u_y, u_y + du_y] \times [u_z, u_z + du_z]$ at a time t . Integrating over the entire phase space, the total number of particles in the system is

$$N_{\text{tot}} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f d^3x d^3u \quad (1.1)$$

Here f is the *number density distribution function*. Alternatively, one could describe the system using the *mass density DF* f_m where $dM = f_m d^3x d^3u$. f and f_m are proportional: $f_m = m f$, where m is the particle mass. The total mass in the system is simply

$$M_{\text{tot}} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_m d^3x d^3u = N_{\text{tot}} m \quad (1.2)$$

Another alternative formulation is to use the *probability density DF* f_p , which expresses the probability of being inside a phase space element $f_p(\mathbf{x}, \mathbf{v}, t)$. This DF is normalised to 1:

$$1 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_p d^3x d^3u \quad (1.3)$$

One can recover the number density DF by multiplying with the total number of particles:

$$f = N_{\text{tot}} f_p(\mathbf{x}, \mathbf{u}, t) \quad (1.4)$$

1.2.2 Moments of the particle DF

Other quantities can be derived by taken in the moments of the particle distribution function. By integrating out the velocity dependence, one gets the zeroth order moment, or the number density

$$n(\mathbf{x}, t) = \int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{u}, t) d^3u \quad (1.5)$$

One can also define the mass density

$$\rho(\mathbf{x}, t) = m n(\mathbf{x}, t) \quad (1.6)$$

The first order moment, the fluid momentum $\mathbf{m}(\mathbf{x}, t)$, is defined as

$$\mathbf{m}(\mathbf{x}, t) = \int_{\mathbb{R}^3} m f(\mathbf{x}, \mathbf{u}, t) \mathbf{u} d^3u \quad (1.7)$$

and is used to define the fluid *bulk* or *macroscopic* velocity $\mathbf{v}(\mathbf{x}, t)$ as

$$\mathbf{m}(\mathbf{x}, t) = \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \quad (1.8)$$

At each position and time, an important new variable can also be defined as the microscopic relative velocity

$$\mathbf{w} = \mathbf{u} - \mathbf{v}(\mathbf{x}, t) \quad (1.9)$$

also known as *thermal* velocity. In general, arbitrary moments can be defined as

$$Q(\mathbf{x}, t) = \int_{\mathbb{R}^3} q(\mathbf{u}, t) f(\mathbf{x}, \mathbf{u}, t) d^3u \quad (1.10)$$

with q a function of the particle velocity. A fundamentally important quantity can finally be defined as the second order moment of the distribution function, this is the fluid total energy:

$$E(\mathbf{x}, t) = \int_{\mathbb{R}^3} \frac{1}{2} m u^2 f(\mathbf{x}, \mathbf{u}, t) d^3u \quad (1.11)$$

Using the thermal velocity and the relation $u^2 = w^2 + 2 \mathbf{w} \cdot \mathbf{v} + v^2$, one obtains immediately the relation

$$E(\mathbf{x}, t) = \frac{1}{2} \rho \sigma_{3D}^2 + \frac{1}{2} \rho v^2 \quad (1.12)$$

where we used the following property of the thermal velocity

$$\int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{u}, t) \mathbf{w} d^3u = 0 \quad (1.13)$$

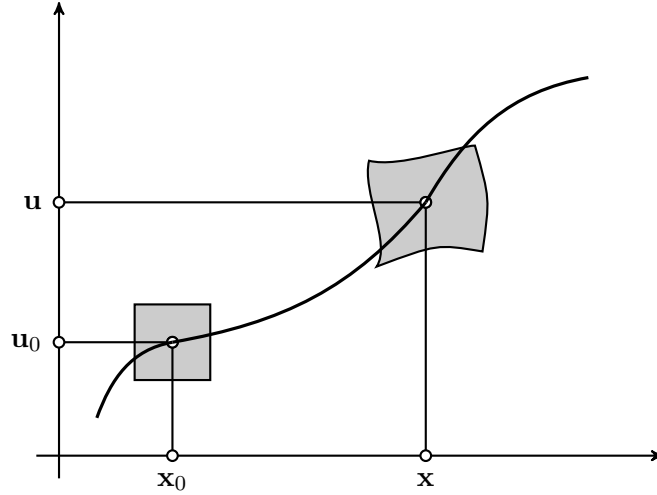


Figure 1.1: A volume element moving along a trajectory in phase space

and we introduced the one dimensional, component-wise microscopic *velocity dispersion*

$$n(\mathbf{x}, t) \sigma_i^2(\mathbf{x}, t) = \int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{u}, t) w_i^2 d^3u \quad (1.14)$$

and the total or three dimensional velocity dispersion as

$$\sigma_{3D}^2(\mathbf{x}, t) = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \quad (1.15)$$

We can now interpret Equation 1.12 as the sum of the fluid thermal energy, which is the kinetic energy associated to microscopic *relative* motions and the bulk kinetic energy, which is the kinetic energy associated to the bulk, average macroscopic flow.

1.3 Boltzmann equation

Elementary mechanics states that a trajectory in phase space is defined to first order by

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{u}_0 dt \quad (1.16)$$

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{a}(\mathbf{x}_0, t_0) dt \quad (1.17)$$

where the position \mathbf{x} and the velocity \mathbf{u} are independent variables. Assume a phase space volume element $d^6V_0 = d^3x_0 d^3u_0$ travels along a trajectory to new a position in phase space (figure 1.1). The new volume element $d^6V = d^3x d^3u$ is related to d^6V_0 through the Jacobian of the coordinate transformation:

$$|\mathbb{J}| = \begin{vmatrix} 1 & dt \\ (\frac{\partial a}{\partial x}) dt & 1 \end{vmatrix} = 1 - \left(\frac{\partial a}{\partial x} \right) dt^2 \quad (1.18)$$

Using $d^6V = |\mathbb{J}| d^6V_0$, we deduce

$$\frac{d^6V - d^6V_0}{dt} \simeq -d^6V_0 \left(\frac{\partial a}{\partial x} \right) dt \rightarrow 0 \text{ when } dt \rightarrow 0 \quad (1.19)$$

In other words, we have

$$\frac{D}{Dt} (d^6V) = 0 \quad (1.20)$$

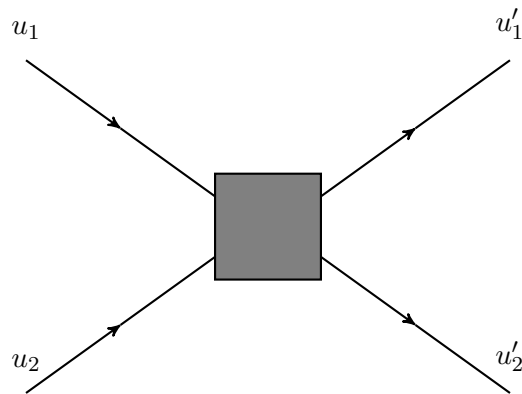


Figure 1.2: Schematic representation of a binary collision

This means the volume element is conserved or $d^6V = d^6V_0$. This result is known as *Liouville's theorem*. If particles cannot be created or destroyed, the particle distribution function is also conserved since

$$dN_0 = f_0 d^3x_0 d^3u_0 \quad (1.21)$$

$$dN = f d^3x d^3u \quad (1.22)$$

leads to

$$f(\mathbf{x}(t_0 + dt), \mathbf{u}(t_0 + dt), t_0 + dt) = f(\mathbf{x}_0, \mathbf{u}_0, t_0) \quad (1.23)$$

This implies that the total time derivative of f is also zero, which can be expressed, using partial derivatives and *the chain rule*, as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial f}{\partial \mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{\partial f}{\partial t} = 0 \quad (1.24)$$

This equation gives the time evolution of the particle distribution function and is also called the Vlasov equation. One can rewrite this as

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{u}} = 0 \quad (1.25)$$

Note that this is a *partial differential equation* in 6-dimensional phase space.

So far, we have not taken into account any collisions. Binary collisions can add or remove particles from a phase space element. The change in particle numbers is expressed by the collision term

$$\left(\frac{Df}{Dt}\right)_{\text{coll}} = \left(\frac{Df}{Dt}\right)_{\text{in}} - \left(\frac{Df}{Dt}\right)_{\text{out}} \quad (1.26)$$

Finally, we get the Boltzmann equation

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{u} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{u}} = \left(\frac{Df}{Dt}\right)_{\text{coll}}} \quad (1.27)$$

1.3.1 Binary collisions

A collision featuring two particles (Fig. 1.2) obeys microscopic conservation laws, which are the conservation of mass, momentum and energy

$$M = m_1 + m_2 = m'_1 + m'_2 \quad (1.28)$$

$$M\mathbf{V} = m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{u}'_1 + m_2\mathbf{u}'_2 \quad (1.29)$$

$$E = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1u'^2_1 + \frac{1}{2}m_2u'^2_2 \quad (1.30)$$

when we evaluate the particle properties before and after the collision, but far enough from the impact point so that the potential energy of the interaction can be ignored. We also consider only elastic collisions for which the mass of the particles before and after the collision remain unchanged so that $m'_1 = m_1$ and $m'_2 = m_2$. At the impact point itself, different kinds of potential energy come into play. For elastic collisions however, the description of these is unimportant and the collision itself can be viewed as a black box. Note that we introduced the velocity of the centre of mass \mathbf{V} and the total mass of the system M . We can also define the relative velocities of the particles (not to be confused with the fluid bulk velocity defined earlier)

$$\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2 \quad (1.31)$$

and the reduced mass

$$\tilde{m} = \frac{m_1m_2}{m_1 + m_2} \quad (1.32)$$

Quite naturally one defines equivalent sets of variable by noticing that

$$\mathbf{u}_1 = \mathbf{V} + \frac{m_2}{m_1 + m_2}\mathbf{v} \quad (1.33)$$

$$\mathbf{u}_2 = \mathbf{V} - \frac{m_1}{m_1 + m_2}\mathbf{v} \quad (1.34)$$

The determinant of the Jacobian matrix of this transformation determines the change in the volume elements

$$|\mathbb{J}| = \begin{vmatrix} 1 & \frac{m_2}{m_1+m_2} \\ 1 & -\frac{m_1}{m_1+m_2} \end{vmatrix} = 1 \quad (1.35)$$

The consequence is that the volume elements between the two sets of variables (\mathbf{v}, \mathbf{V}) and $(\mathbf{u}_1, \mathbf{u}_2)$ are equal.

$$d^3Vd^3v = d^3u_1d^3u_2 \quad (1.36)$$

This relation is also valid for $(\mathbf{u}'_1, \mathbf{u}'_2)$ and $(\mathbf{v}', \mathbf{V})$. Using these new variables, the conservation of the total energy can also be written as

$$E = \frac{1}{2}MV^2 + \frac{1}{2}\tilde{m}v^2 = \frac{1}{2}MV^2 + \frac{1}{2}\tilde{m}v'^2 \quad (1.37)$$

resulting in $\|\mathbf{v}\| = \|\mathbf{v}'\|$ or in short $v = v'$. This is a fundamental result of the theory of binary collisions: the magnitude of the relative velocity does not change before and after the collision, only the direction of the relative velocity will change. In other words, the transformation between \mathbf{v} and \mathbf{v}' is a rotation, which also preserves the volume element, so that $d^3v = d^3v'$. Combined with the previous relations, one finally gets

$$d^3u'_1d^3u'_2 = d^3u_1d^3u_2, \quad (1.38)$$

a result that will prove very usefull later.

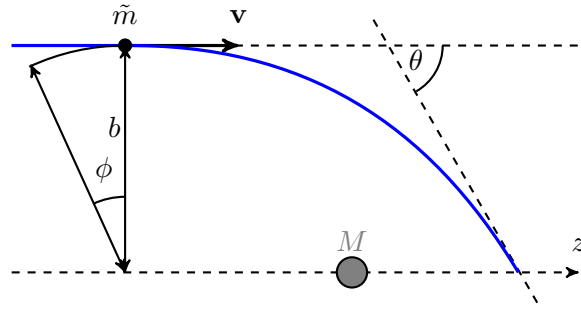


Figure 1.3: Definition of the impact parameter

1.3.2 Cross section and collision rate

Now, we want to compute the collision rates for both outgoing and incoming collisions. We will start with outgoing collisions. We consider as targets all particle of type 1 with mass m_1 and velocity \mathbf{u}_1 and as collision partners all particle of type 2 with mass m_2 and velocity \mathbf{u}_2 . We will use the new variables M , \mathbf{V} and \tilde{m} , \mathbf{v} and put ourselves in the frame of the centre of mass, so that only the relative velocities \mathbf{v} and \mathbf{v}' matter.

Each particle of type 1 will be bombarded by particles of type 2 with impact parameter b , relative velocity \mathbf{v} and collision plane angle ϕ . The incoming relative velocity is assumed to be aligned with the z axis in Figure 1.3. At infinity, before the collision, particles of type 2 will come from the left, at a constant relative speed. As soon as they come close enough to their target particle of type 1, they will start to “feel” the interaction with the other particle.

Typical interactions we consider here are atomic or molecular interactions, that are modelled using Coulomb forces or a simple bouncing sphere model. A key property of these interactions is that the interparticle force depends only on the relative distance between the two particles. As a consequence, as you learned from classical mechanics, the angular momentum is conserved and the orbits of the two particles will remain in the same plane, called the collision plane. This plane is defined by the collision plane angle ϕ , which remains constant during the collision so that $\phi' = \phi$.

For a given impact parameter b , the outcome of the collision is a particle of type 2 which is deflected by the interaction with a deflection angle θ , so that after the collision, the particle is leaving to the right at constant speed $v' = v$ along this deflection angle and to infinity.

For a given time step dt during the interaction, we count how many particles of type 2 we have flowing passed the target particle of type 1. For this, we will use the previously defined distribution function,

$$dN_2 = f_2 dV d^3u_2 \quad (1.39)$$

where the volume element we need to consider is the cylinder swept by the collision partners during the time step and in the surface defined by two infinitesimally close impact parameters b and $b + db$ and collision planes ϕ and $\phi + d\phi$,

$$dV = b db d\phi v dt \quad (1.40)$$

The total number of collisions is then obtained by multiplying by the number of targets in the phase space volume element,

$$dN_{\text{out}} = dN_2 f_1 d^3x d^3u_1 = f_1 f_2 v b db d\phi d^3u_1 d^3u_2 d^3x dt \quad (1.41)$$

To obtain the final collision rate, we need to integrate this infinitesimal contribution over all possible impact parameter, over all possible collisional planes and all possible type 2 particle

velocities, dividing by the phase-space volume element for particle type 1 and by the time step,

$$\left(\frac{Df}{Dt}\right)_{\text{out}} = \frac{dN_{\text{out}}}{d^3x d^3u_1 dt} = \int_0^{+\infty} \int_0^{2\pi} \int_{\mathbb{R}^3} f_1 f_2 v b db d\phi d^3u_2 \quad (1.42)$$

For the sake of simplicity, a new variable is introduced, called the collision *cross section*, and is defined so that $b db = \sigma \sin \theta d\theta$, or equivalently

$$\sigma(\mathbf{u}_1, \mathbf{u}_2) = \frac{b}{\sin \theta} \left(\frac{\partial b}{\partial \theta}\right) \quad (1.43)$$

The cross section has the units of a surface, and can be computed directly from the relation connecting the deflection angle θ to the impact parameter b for a given type of interaction. The cross section can be interpreted as an effective area for the target particles of type 1 as seen from the collision partners of type 2. Using this new variable and combining the deflection angle θ to the collision plane angle ϕ , one finally obtains

$$\left(\frac{Df}{Dt}\right)_{\text{out}} = \int_{4\pi} \int_{\mathbb{R}^3} f_1 f_2 \sigma v d\Omega d^3u_2 \quad (1.44)$$

where the *solid angle* Ω is classically defined as $d\Omega = \sin \theta d\theta d\phi$, but has nothing to do with the spherical coordinates. It is a compact notation that describes the geometry of the collisions.

In astrophysics, we use usually two different cross-sections: the first one, called “hard sphere” cross-section, assumes that particles are like billiard balls. This is the regime relevant for molecular or neutral atomic gases, for which particles do not “feel” each other unless their respective electronic clouds interact with each other. It is possible, using pure geometric arguments, to compute the cross-section in this simplified case (left for the reader as an exercise)

$$\sigma = r_0^2 \quad (1.45)$$

where r_0 is the radius of the sphere. In this molecular regime, one usually consider for this constant cross section the value $\sigma_0 \simeq 10^{-15} \text{cm}^2$. The second important regime in astrophysics is for an ionised gas, made of electrons and protons mostly. These conditions are usually called a plasma. Charged particles can interact at long distances through the Coulomb interaction. In this case, it is also possible to compute the cross section (also left to the reader as an exercise) and one obtains the famous Rutherford formula

$$\sigma = \left(\frac{e^2}{\tilde{m}v^2}\right)^2 \frac{1}{(\sin \frac{\theta}{2})^4} \quad (1.46)$$

where e is the electron charge, \tilde{m} the reduced mass, v the relative velocity and θ the deflection angle. One can see that the cross-section diverges for very small deflection angles. This limit corresponds to particles with very large impact parameters, whose contribution to the collision integral diverges. The usual approach is to ignore particles with impact parameters larger than the Debye length, beyond which the effective charge of the plasma is zero and Coulomb interactions vanish. We will come back to this point later in the course.

1.3.3 Collision integral

In order to derive the Boltzmann equation, we need to compute the collision rate also for incoming collisions. These collisions can be described as a transition in velocity space from

particles labelled with “prime” before the collisions and labelled without “prime” after the collisions. We can therefore directly obtain the number of incoming collisions as

$$dN_{\text{in}} = f'_1 f'_2 \sigma' v' d\Omega' d^3 u'_1 d^3 u'_2 d^3 x dt \quad (1.47)$$

We already know from the previous section that the magnitude of the relative velocity is constant $v' = v$ and that $d^3 u'_1 d^3 u'_2 = d^3 u_1 d^3 u_2$. Moreover, the reverse collisions we consider here have the same cross section and the same geometry $d\Omega' = d\Omega$ as the direct collisions we have considered in the previous section. This reversibility is a key property of Hamiltonian systems with a central force $F(r)$ and can be expressed as $\sigma' = \sigma(\mathbf{u}'_1, \mathbf{u}'_2) = \sigma(\mathbf{u}_1, \mathbf{u}_2) = \sigma$. Another property of such collisions is the invariance when switching particle 1 and 2, so that $\sigma(\mathbf{u}_1, \mathbf{u}_2) = \sigma(\mathbf{u}_2, \mathbf{u}_1)$. We thus can rewrite the number of incoming collisions as

$$dN_{\text{in}} = f'_1 f'_2 \sigma v d\Omega d^3 u_1 d^3 u_2 d^3 x dt \quad (1.48)$$

Dividing by the phase-space volume element $d^3 x d^3 u_1$ and the time step dt , and integrating over all possible velocities and solid angles, we obtain

$$\left(\frac{Df}{Dt}\right)_{\text{in}} = \int_{4\pi} \int_{\mathbb{R}^3} f'_1 f'_2 \sigma v d\Omega d^3 u_2 \quad (1.49)$$

We can add this incoming collision rate to the outgoing one in the right-hand side of the Boltzmann equation and obtain its final form

$$\boxed{\frac{\partial f_1}{\partial t} + \mathbf{u}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f_1}{\partial \mathbf{u}_1} = \int_{4\pi} \int_{\mathbb{R}^3} (f'_1 f'_2 - f_1 f_2) \sigma v d\Omega d^3 u_2} \quad (1.50)$$

Traditionally, the index 1 is dropped from all variables in the left-hand side of the Boltzmann equation, but we keep it here for sake of clarity. The right-hand side is called the collision integral. The microscopic physics is fully encoded in the actual form of the cross section and determines the magnitude of the collision integral. The Boltzmann equation is a rather complex integro-differential equation, but once we know $f(\mathbf{x}, \mathbf{u}_1, t)$ everywhere in phase-space, we can compute, for each collision partner \mathbf{u}_2 and for each solid angle Ω , the primed velocities \mathbf{u}'_1 and \mathbf{u}'_2 using the adopted collision geometry and the microscopic conservation laws. Using $f'_1 = f(\mathbf{x}, \mathbf{u}'_1, t)$ and $f'_2 = f(\mathbf{x}, \mathbf{u}'_2, t)$, we can compute the collision integral, a rather daunting task, and update f at the next time step.

1.3.4 Collision invariants

We have already used moments of the DF to introduce macroscopic quantities such as the mass density $\rho(\mathbf{x}, t)$, the bulk fluid velocity $\mathbf{v}(\mathbf{x}, t)$ and the fluid total energy $E(\mathbf{x}, t)$. In the next sections, we will go one step further and take the moments of the Boltzmann equation itself, deriving macroscopic conservation laws describing the dynamics of the macroscopic fluid variables. Note that this will require to perform the rather scary exercise of taking the moments of the collision integral itself. But, as we will see now, under certain conditions, this turns out to be surprisingly simple.

We define our generalized moment using the following macroscopic quantity

$$I(\mathbf{x}, t) = \int_{\mathbb{R}^3} \int_{4\pi} \int_{\mathbb{R}^3} Q(\mathbf{u}_1) (f'_1 f'_2 - f_1 f_2) \sigma v d\Omega d^3 u_2 d^3 u_1 \quad (1.51)$$

The function $Q(\mathbf{u}_1)$ is for now an arbitrary function of the particle velocity and defines the type of moment we would like to compute. Note that this is a triple integral, and variable $\mathbf{u}_1, \mathbf{u}_2$,

Ω are all dummy integration variables. We will now switch the indice 1 and 2 in the previous expression. As we discussed in the previous section, the collision properties are unchanged under this rather trivial transformation. So we have

$$I(\mathbf{x}, t) = \int_{\mathbb{R}^3} \int_{4\pi} \int_{\mathbb{R}^3} Q(\mathbf{u}_2)(f'_2 f'_1 - f_2 f_1) \sigma v \, d\Omega \, d^3 u_1 \, d^3 u_2 \quad (1.52)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} \int_{4\pi} \int_{\mathbb{R}^3} [Q(\mathbf{u}_1) + Q(\mathbf{u}_2)] (f'_1 f'_2 - f_1 f_2) \sigma v \, d\Omega \, d^3 u_1 \, d^3 u_2 \quad (1.53)$$

where we use in the second equality the clever trick that if two expressions are equal, they are also equal to their average. In this new expression, we now switch the prime and the non-prime velocity variables, to obtain

$$I(\mathbf{x}, t) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{4\pi} \int_{\mathbb{R}^3} [Q(\mathbf{u}'_1) + Q(\mathbf{u}'_2)] (f_1 f_2 - f'_1 f'_2) \sigma' v' \, d\Omega' \, d^3 u'_1 \, d^3 u'_2 \quad (1.54)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} \int_{4\pi} \int_{\mathbb{R}^3} [-Q(\mathbf{u}'_1) - Q(\mathbf{u}'_2)] (f'_1 f'_2 - f_1 f_2) \sigma v \, d\Omega \, d^3 u_1 \, d^3 u_2 \quad (1.55)$$

where we derived the second line using the reversibility of the collisions and the conservation laws leading to $v' = v$ and $d^3 u'_1 d^3 u'_2 = d^3 u_1 d^3 u_2$. We also put a minus sign in the square bracket to recover the classical collision integral form. Using the same trick as before, namely adding the two (equal) expressions and dividing by two, we finally obtain the fundamental result

$$I(\mathbf{x}, t) = \frac{1}{4} \int_{\mathbb{R}^3} \int_{4\pi} \int_{\mathbb{R}^3} [Q(\mathbf{u}_1) + Q(\mathbf{u}_2) - Q(\mathbf{u}'_1) - Q(\mathbf{u}'_2)] (f'_1 f'_2 - f_1 f_2) \sigma v \, d\Omega \, d^3 u_1 \, d^3 u_2 \quad (1.56)$$

The still unspecified quantity Q is said to be a *collision invariant* if it satisfies the following conservation law

$$Q(\mathbf{u}_1) + Q(\mathbf{u}_2) = Q(\mathbf{u}'_1) + Q(\mathbf{u}'_2). \quad (1.57)$$

In this case, according to the previous derivation, its moment vanishes identically so that $I(\mathbf{x}, t) = 0$ everywhere. Why is this result so powerful? We already know three collisional invariants from the microscopic conservation laws for the mass $Q(\mathbf{u}_1) = m_1$, the momentum $Q(\mathbf{u}_1) = m_1 \mathbf{u}_1$ and the kinetic energy $Q(\mathbf{u}_1) = \frac{1}{2} m_1 u_1^2$. This will allow us to take the corresponding moments for the Boltzmann equation and the right-hand side, associated to the particularly scary collision integral, will vanish. We will only have to deal with the left-hand side, namely to take the moments of the Vlasov equation, a much simpler task that will occupy us in a future chapter.

1.4 Local Thermodynamical Equilibrium

In this section, we will discuss a particular case of the Boltzmann equation, for which the collision integral vanishes exactly. This case is of particular importance, as it will lead us to the derivation of the Euler equations and the theoretical foundation of astrophysical fluid dynamics. We will use for the first time in this course the notion of *equilibrium solution*. The collision integral is the difference between two terms: a source of incoming particles and a sink of outgoing particles. Under the proper conditions (we will derive the exact requirement later), both the source and the sink terms can be very large, much larger than all the terms in the left-hand side of the Boltzmann equation. In this case, the system will naturally evolve to a state where the two collision terms exactly cancel each other, leading to the notion of *detailed balance* and more fundamentally to *thermodynamical equilibrium*. There is a solution to these equilibrium conditions, called the Maxwell-Boltzmann DF and usually noted f_0 .

1.4.1 Maxwell-Boltzmann distribution

The derivation of the Maxwell-Boltzmann equilibrium DF is the result of requiring detailed balance of incoming and outgoing collisions in each phase-space element. This is a very strong condition: one could imagine that the collision term vanishes in a global, weaker sense, after performing the integral over velocity space. Detailed balance leads to the equilibrium DF $f_0(\mathbf{x}, \mathbf{v}, t)$ which satisfies $\left(\frac{Df}{Dt}\right)_{\text{coll}} \equiv 0$ in a strong sense, namely that

$$\left(\frac{Df}{Dt}\right)_{\text{coll}} = \int_{4\pi} \int_{\mathbb{R}^3} \underbrace{(f'_1 f'_2 - f_1 f_2)}_{\equiv 0} \sigma v d\Omega d^3 u_2 \quad (1.58)$$

so the integrand vanishes everywhere in velocity space, at a given spatial location. This detailed balance of incoming and outgoing collisions defines what is called *local thermodynamic equilibrium* (LTE). We can reformulate this condition as

$$f_1 f_2 = f'_1 f'_2 \quad (1.59)$$

$$\text{or } \ln f_1 + \ln f_2 = \ln f'_1 + \ln f'_2 \quad (1.60)$$

We see that detailed balance implies that $\ln f_0$ is a collision invariant, in addition to the microscopic conservation laws we have already introduced.

$$m_1 + m_2 = m'_1 + m'_2 \quad (1.61)$$

$$m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2 = m_1 \mathbf{u}'_1 + m_2 \mathbf{u}'_2 \quad (1.62)$$

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 \quad (1.63)$$

We have also seen that these three conservation laws are enough, together with the cross section σ , to fully specify the properties of the collisions. We cannot have a fourth collision invariant that brings in new information to the phase-space transform induced by collisions. The only possibility is that this fourth invariant is a linear combination of the previous three, so that

$$\ln f_0 = \alpha m + \beta \frac{1}{2} m u^2 + \boldsymbol{\gamma} \cdot m \mathbf{u} \quad (1.64)$$

where $\alpha(\mathbf{x}, t)$ and $\beta(\mathbf{x}, t)$ are scalar quantities that depend on space and time and $\boldsymbol{\gamma}(\mathbf{x}, t)$ is a vector quantity that depends also on space and time. These quantities are considered as constant in velocity space for a fixed spatial location, but they are allowed to vary in physical

space. They are macroscopic properties of the fluid, and can be related to the moments of the Maxwell-Boltzmann DF as follows

$$\int_{\mathbb{R}^3} f_0(\mathbf{u}) d^3u = n(\mathbf{x}, t) \quad (1.65)$$

$$\int_{\mathbb{R}^3} f_0(\mathbf{u}) \mathbf{u} d^3u = n(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \quad (1.66)$$

$$\int_{\mathbb{R}^3} f_0(\mathbf{u}) (\mathbf{u} - \mathbf{v})^2 d^3u = n(\mathbf{x}, t) \sigma_{3D}^2(\mathbf{x}, t) \quad (1.67)$$

One key property is that this DF is isotropic in velocity space: it depends only on $u^2 = u_x^2 + u_y^2 + u_z^2$ so that no direction is preferred. As a consequence, the velocity dispersion satisfies

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \sigma_{1D}^2 = \frac{1}{3} \sigma_{3D}^2 \quad (1.68)$$

so we can drop the subscripts and use only $\sigma(\mathbf{x}, t)$ for the 1D velocity dispersion. Inverting the previous three moments (left to the reader as an exercise), one obtains the following closed form for the Maxwell-Boltzmann DF

$$\boxed{f_0(\mathbf{x}, \mathbf{u}, t) = \frac{n(\mathbf{x}, t)}{(2\pi\sigma(\mathbf{x}, t)^2)^{3/2}} \exp\left(-\frac{1}{2} \frac{(\mathbf{u} - \mathbf{v}(\mathbf{x}, t))^2}{\sigma(\mathbf{x}, t)^2}\right)} \quad (1.69)$$

It is quite convenient to introduce the one-dimensional Gaussian distribution, with zero mean and variance σ

$$G(w) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{w^2}{2\sigma^2}\right) \quad (1.70)$$

and define the three-dimensional Maxwell-Boltzmann distribution as the product of three Gaussian

$$f_0(\mathbf{x}, \mathbf{u}, t) = n(\mathbf{x}, t) G(u_x - v_x) G(u_y - v_y) G(u_z - v_z) \quad (1.71)$$

The arguments of the Gaussian function have been defined earlier: these are the components of the thermal velocity $\mathbf{w} = \mathbf{u} - \mathbf{v}$. The one-dimensional Gaussian distribution has the following useful properties

$$\int_{-\infty}^{+\infty} G(w) dw = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} w G(w) dw = 0, \quad (1.72)$$

$$\int_{-\infty}^{+\infty} w^2 G(w) dw = \sigma^2, \quad \int_{-\infty}^{+\infty} w^3 G(w) dw = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} w^4 G(w) dw = 3\sigma^4 \quad (1.73)$$

We will see later that the second-order moment of the Maxwell-Boltzmann DF can be identified to the pressure of an ideal gas, for which we have

$$P = n k_B T = \rho \sigma^2 \quad (1.74)$$

As a consequence, we obtain

$$\sigma(\mathbf{x}, t) = \sqrt{\frac{k_B T(\mathbf{x}, t)}{m}}. \quad (1.75)$$

The velocity dispersion is thus directly related to the gas temperature. The temperature is the traditional quantity introduced in statistical mechanics to define thermodynamical equilibrium.

1.4.2 Boltzmann's H theorem

In the previous section, we have seen that a sufficient condition for thermodynamical equilibrium is that the DF is equal to the Maxwell-Boltzmann distribution. But it is not clear yet whether any system initially out of equilibrium will indeed necessarily relax towards the Maxwell-Boltzmann DF. This important result is the conclusion of the famous Boltzmann's H theorem. In order to simplify the discussion, we consider a uniform medium without any external acceleration. In this case, we can drop all \mathbf{x} dependent variables in the Boltzmann equation, which now contains only the partial time derivative in the left-hand side and the collision integral in the right-hand side.

$$\frac{\partial f}{\partial t} = \int_{\mathbb{R}^3} \int_{4\pi} (f'_1 f'_2 - f_1 f_2) \sigma v \, d\Omega \, d^3 u_2 \quad (1.76)$$

We now introduce a new moment of the DF, the quantity

$$H(t) = \int_{\mathbb{R}^3} f(u, t) \ln(f(u, t)) \, d^3 u \quad (1.77)$$

where f is the *out-of-equilibrium* DF. This quantity is the opposite of the entropy defined in statistical mechanics. Taking the time derivative, one obtains

$$\frac{\partial H}{\partial t} = \int_{\mathbb{R}^3} \left[\frac{\partial f}{\partial t} \ln f + f \frac{\partial}{\partial t} \ln f \right] d^3 u = \int_{\mathbb{R}^3} [\ln f + 1] \frac{\partial f}{\partial t} d^3 u \quad (1.78)$$

We can now replace the time derivative by the collision integral and obtain a very familiar form

$$\frac{\partial H}{\partial t} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{4\pi} [\ln f_1 + 1] (f'_1 f'_2 - f_1 f_2) \sigma v \, d\Omega \, d^3 u_1 \, d^3 u_2 \quad (1.79)$$

$$= \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{4\pi} [\ln f_1 + \ln f_2 - \ln f'_1 - \ln f'_2] (f'_1 f'_2 - f_1 f_2) \sigma v \, d\Omega \, d^3 u_1 \, d^3 u_2 \quad (1.80)$$

where we apply the results of Section 1.3.4 to derive the second line. We finally rearrange the integral into

$$\boxed{\frac{\partial H}{\partial t} = -\frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{4\pi} [\ln f'_1 f'_2 - \ln f_1 f_2] (f'_1 f'_2 - f_1 f_2) \sigma v \, d\Omega \, d^3 u_1 \, d^3 u_2} \quad (1.81)$$

The logarithm being a growing function of its argument, if $X > Y$ (resp. $X < Y$), one has $\ln X > \ln Y$ (resp. $\ln X < \ln Y$), so that $(\ln X - \ln Y)(X - Y)$ is always a positive quantity. As a consequence, the time derivative of H is always negative and H will necessarily decrease with time. For a system with finite volume or finite mass, the function will quickly reach a minimum value for which the time derivative is zero and the DF is equal to the Maxwell-Boltzmann one.

This theorem has profound consequences for the physics of gases: it introduces the notion of dissipation as a fundamental property of fluid dynamics. It also leads to a paradox: why do microscopic collisions with purely reversible equation of motions (Hamiltonian systems) lead to a macroscopic dissipative system? One interpretation (among many others) is based on molecular chaos: even Hamiltonian systems are unpredictable on sufficiently long time scales and are consistent with a dissipative, irreversible evolution.

1.4.3 Collision time and mean free path

A legitimate question follows: how much time does the system need to reach equilibrium? In kinetic theory, many different time scales can be computed to estimate this relaxation time: the *stopping time* corresponds to the time required for a particle to loose its initial kinetic

energy through collisions, the *deflection time* corresponds to the time required for a particle to be deflected by 90° from its initial trajectory, etc. These different times are all slightly different estimates of what is usually refer to as the collision time. For sake of simplicity, we will compute it using the rate of outgoing collisions, namely the negative term in the collision integral, assuming that the DF is the Maxwell-Boltzmann one and the collision cross section is for “hard sphere” particles, typical for a molecular or neutral atomic gas.

We define the collision rate as the following zeroth-order moment of the collision integral

$$C_{\text{out}} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{4\pi} f_1 f_2 \sigma v d^3 u_1 d^3 u_2 d\Omega \quad (1.82)$$

where $f_1 = f_0(u_1)$, $f_2 = f_0(u_2)$ and $\sigma = r_0^2$ for hard spheres. We can easily perform the integration over the solid angle

$$\int_{4\pi} \sigma d\Omega = 4\pi r_0^2 \quad (1.83)$$

and we are left with a double integral over velocity space

$$C_{\text{out}} = 4\pi r_0^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_0(u_1) f_0(u_2) v d^3 u_1 d^3 u_2 \quad (1.84)$$

Changing variables from $(\mathbf{u}_1, \mathbf{u}_2)$ to (\mathbf{V}, \mathbf{v}) , we get (left to the reader as an exercise)

$$C_{\text{out}} = 8r_0^2 n^2 \left(\frac{m}{k_B T} \right)^3 \int_0^{+\infty} e^{-\frac{1}{2} \frac{MV^2}{k_B T}} V^2 dV \int_0^{+\infty} e^{-\frac{1}{2} \frac{mv^2}{k_B T}} v^3 dv \quad (1.85)$$

which can be finally integrated as

$$C_{\text{out}} = n^2 4\sqrt{\pi} r_0^2 \sqrt{\frac{k_B T}{m}} \quad (1.86)$$

In this derivation, we see that the collision rate is proportional to the square of the fluid density, to the cross section and to the particle velocity dispersion. In the general case, one usually prefers the notation

$$C_{\text{out}} = n^2 \langle \sigma v \rangle \quad (1.87)$$

where the angle brackets denote the average over velocity space. For a constant cross section σ_0 , one can quickly derive the approximate formula

$$\boxed{C_{\text{out}} \simeq n^2 \sigma_0 \sqrt{\frac{k_B T}{m}}} \quad (1.88)$$

which is reasonably accurate for hard spheres. For Coulomb interactions, however, computing the collision rate is much more challenging. We present here only an approximate derivation. The idea is to compute an estimate of the cross section as the square of the typical impact parameter. For a Coulomb interaction, the typical impact parameter can be estimated by considering marginally bound orbits, for which (using for the reduced mass the mass of the electron)

$$\frac{e^2}{b} \simeq \frac{1}{2} m_e v^2 \quad (1.89)$$

which gives for the cross section (dropping all dimensionless numbers that are close to unity)

$$\sigma \simeq \pi b^2 \simeq \left(\frac{e^2}{m_e v^2} \right)^2 = \left(\frac{e^2}{k_B T} \right)^2 \quad (1.90)$$

In the last formula, we replace the relative velocity by the velocity dispersion of electrons, which gives us an average cross section that we identified with σ_0 in the previous formulae, resulting in the following approximation for the collision rate in the Coulomb case

$$C_{\text{out}} \simeq n^2 \frac{e^4}{(k_B T)^2} \sqrt{\frac{k_B T}{m_e}} \quad (1.91)$$

One last important ingredient is missing, namely the fact that the Coulomb cross section is diverging for very large impact parameter. This results in a significant increase of the effective cross section, encoded in what is called the Coulomb logarithm. It is equal to the logarithm of the ratio of the maximum impact parameter, taken equal to the Debye length, and the minimum impact parameter, usually equal to the 90° deflection angle impact parameter. In short, we have

$$\ln \Lambda = \ln \frac{b_{\text{max}}}{b_{\text{min}}} \quad (1.92)$$

and the final (surprisingly accurate) formula for the Coulomb interaction collision rate is

$$C_{\text{out}} \simeq n^2 \frac{e^4}{(k_B T)^2} \sqrt{\frac{k_B T}{m}} \ln \Lambda \quad (1.93)$$

Because of its logarithmic dependence, the Coulomb logarithm takes values $\ln \Lambda \simeq 10$ to 30 for a wide range of astrophysical conditions. The best strategy is to assume it is constant with $\ln \Lambda = 20$.

In order to derive the collision time scale, one just divides the particle density by the collision rate

$$\tau_{\text{coll}} = \frac{n}{C_{\text{out}}} = \frac{1}{n \langle \sigma v \rangle} \quad (1.94)$$

One can also define the *mean free path* as the typical distance a particle will travel within one collision time. Since the typical particle velocity is given by the thermal velocity, one obtains

$$\lambda_{\text{coll}} = \sqrt{\frac{k_B T}{m}} \tau_{\text{coll}} \quad (1.95)$$

For hard spheres, one sees that the mean free path depends only on the inverse of the density, and not on temperature. For ionized plasma's, on the other hand, it depends also on the square of the temperature.

1.4.4 Validity of LTE approximations

Equipped with these notions of mean free path and relaxation time, we can now go back to the Boltzmann equation

$$\frac{\partial f_1}{\partial t} + \mathbf{u} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \mathbf{a}(\mathbf{x}) \cdot \frac{\partial f_1}{\partial \mathbf{u}} = \int_{\mathbb{R}^3} \int_{4\pi} [f_1' f_2' - f_1 f_2] \sigma v \, d^3 u_2 \, d\Omega \quad (1.96)$$

If the right-hand side dominates over the left-hand side, collisions will quickly drive the DF towards the Maxwell-Boltzmann one, validating the LTE approximation. On the other-hand, if the left-hand side dominates, then the LTE approximation won't be valid anymore and the DF will deviate strongly from f_0 . Most of the important terms in the Vlasov equation are spatial gradients of f . If one assume that $f \simeq f_0$, at least initially, we can estimate the magnitude of the gradient terms (left to the reader as an exercise) using the following length scales

$$\frac{1}{h_n} = \frac{|\nabla n|}{n} \quad \text{or} \quad \frac{1}{h_T} = \frac{|\nabla T|}{T} \quad (1.97)$$

and compare them the mean free path as an estimate of the magnitude of the collision integral. The Maxwell-Boltzmann DF, or the LTE approximation are therefore valid if the typical length scale over which the macroscopic variables vary are much larger than the mean free path or

$$\min(h_n, h_T, h_v) \gg \lambda_{\text{coll}} \quad (1.98)$$

It is now your responsibility to check *a posteriori*, after you have found a solution of the fluid equations for your favorite astrophysical objects, that this solution satisfies the validity criterion for LTE. If not, then you cannot use the fluid equation in the LTE regime, and you have to use more complex equations that we will derive in the next section.

1.5 Moments of the Boltzmann equation and the fluid equations

In this section, we will derive the fluid equations, which are the macroscopic version of the microscopic conservation laws. The key property we will use, is the fact that moments of the collision integral vanish if we use collision invariants. This property is true for any DF, even far from LTE. On the other hand, when the system is close to LTE, the fluid equations simplify even more and we obtain the famous Euler equations. Of course, these are valid only if LTE conditions are met, as explained in the previous section.

1.5.1 General case for non-LTE conditions

We now take the moments of the entire Boltzmann equation, by performing the integral over velocity space

$$\int_{\mathbb{R}^3} Q(\mathbf{u}_1) \left(\frac{\partial f_1}{\partial t} + \mathbf{u} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \mathbf{a}(\mathbf{x}) \cdot \frac{\partial f_1}{\partial \mathbf{u}} \right) d^3 u_1 = \int_{\mathbb{R}^3} Q(\mathbf{u}_1) \left(\frac{Df}{Dt} \right)_{\text{coll}} d^3 u_1 \quad (1.99)$$

We have demonstrated earlier that if we use for Q any of m , mu_x , mu_y , mu_z or $\frac{1}{2}mu^2$, in other words any of the microscopic conservation laws, the right-hand side with the collision integral vanishes everywhere. So we just have to take the moments of the left-hand side, namely the Vlasov equation. This is what we will do now, using each microscopic conservation law in sequence.

Mass conservation

This is the simplest of the 3 fluid equations. Multiplying by the particle mass m , we obtain 3 terms

$$\int_{\mathbb{R}^3} m \frac{\partial f}{\partial t} d^3 u + \int_{\mathbb{R}^3} m \mathbf{u} \cdot \frac{\partial f}{\partial \mathbf{x}} d^3 u + \int_{\mathbb{R}^3} m \mathbf{a}(\mathbf{x}) \cdot \frac{\partial f}{\partial \mathbf{u}} d^3 u = 0 \quad (1.100)$$

(1) (2) (3)

In order to compute these 3 terms, the key argument is that \mathbf{x} , \mathbf{u} and t are three independent variables. Let's work out each term separately. For term (1), we can take the time derivative outside of the integral and recover immediately the zeroth-order moment of the DF

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} m f d^3 u = \frac{\partial}{\partial t} \rho(\mathbf{x}, t) \quad (1.101)$$

The second term (2) requires to introduce a famous vector calculus relation

$$\nabla \cdot (f \mathbf{u}) = f \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla f \quad (1.102)$$

where we have noted ∇ the operator $\frac{\partial}{\partial \mathbf{x}}$. Because \mathbf{u} and \mathbf{x} are independant variables, $\nabla \cdot \mathbf{u} = 0$ in this relation. Term (2) can then be simplified as

$$\nabla \cdot \int_{\mathbb{R}^3} m \mathbf{u} f d^3 u = \nabla \cdot (\rho \mathbf{v}) \quad (1.103)$$

where we recognize in the right-hand side the first-order moment of the DF, namely the macroscopic momentum. The third term is more involving. We first need to decompose the dot product into three additional terms

$$m \mathbf{a}(\mathbf{x}) \cdot \int_{\mathbb{R}^3} \frac{\partial f}{\partial \mathbf{u}} d^3 u = m a_x \int_{\mathbb{R}^3} \frac{\partial f}{\partial u_x} d^3 u + m a_y \int_{\mathbb{R}^3} \frac{\partial f}{\partial u_y} d^3 u + m a_z \int_{\mathbb{R}^3} \frac{\partial f}{\partial u_z} d^3 u \quad (1.104)$$

Let's deal with the first one. For this, we introduce the one-dimensional distribution function as

$$F(u_x) = \int_{\mathbb{R}^2} f du_y du_z \quad \text{for which we have} \quad \int_{-\infty}^{+\infty} \frac{\partial F}{\partial u_x} du_x = \int_{\mathbb{R}^3} \frac{\partial f}{\partial u_x} d^3 u \quad (1.105)$$

We see that this function can be directly integrated so that

$$m a_x \int_{\mathbb{R}^3} \frac{\partial f}{\partial u_x} d^3 u = m a_x [F(+\infty) - F(-\infty)] = 0 \quad (1.106)$$

since $F \rightarrow 0$ when $u_x \rightarrow \pm\infty$, a required property for distribution function. The same trick can be used for each direction. The final result is the equation for mass conservation, the first of our macroscopic fluid equations.

$$\boxed{\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) = 0} \quad (1.107)$$

Momentum conservation

We now use for our moment's calculation the quantity $Q = m u_x$. We will drop the index x later when it will become clearer. The integration over the Boltzmann equation now becomes

$$\int_{\mathbb{R}^3} m u_x \frac{\partial f}{\partial t} d^3 u + \int_{\mathbb{R}^3} m u_x \mathbf{u} \cdot \nabla f d^3 u + \int_{\mathbb{R}^3} m u_x \mathbf{a}(\mathbf{x}) \cdot \frac{\partial f}{\partial \mathbf{u}} d^3 u = 0 \quad (1.108)$$

(1) (2) (3)

with again three main terms to integrate. The first term (1) is again quite easy because time and velocity commute.

$$(1) = \frac{\partial}{\partial t} \int_{\mathbb{R}^3} m u_x f d^3 u = \frac{\partial}{\partial t} (\rho v_x) \quad (1.109)$$

The third term can be decomposed in three more terms by developing the dot product

$$(3) = m a_x \int_{\mathbb{R}^3} u_x \frac{\partial f}{\partial u_x} d^3 u + m a_y \int_{\mathbb{R}^3} u_x \frac{\partial f}{\partial u_y} d^3 u + m a_z \int_{\mathbb{R}^3} u_x \frac{\partial f}{\partial u_z} d^3 u \quad (1.110)$$

(3a) (3b) (3c)

The last two terms vanish exactly. One can define, almost like before, but not quite

$$H(u_y) = \int_{\mathbb{R}^2} u_x f du_x du_z \quad (1.111)$$

so that the second term (3b) can be written as

$$(3b) = m a_y \int_{-\infty}^{+\infty} \frac{\partial H}{\partial u_y} du_y = m a_y [H(+\infty) - H(-\infty)] = 0 \quad (1.112)$$

The same result applies to the third term (3c). The term (3a), on the other hand, does not vanish. We define, now exactly like before

$$F(u_x) = \int_{\mathbb{R}^2} f du_y du_z \quad (1.113)$$

so that the third term (3) comes only from (3a) that can be integrated by parts

$$(3) = a_x \int_{-\infty}^{+\infty} m u_x \frac{\partial F}{\partial u_x} du_x = a_x \left([m u_x F(u_x)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} m F(u_x) du_x \right) = -\rho a_x \quad (1.114)$$

The function F inside the brackets is evaluated at infinity, but since it is a distribution function, it converges to zero at infinity faster than u_x , so that the bracket vanishes. The remaining integral with a minus sign is just equal to $-\rho$.

Let's now focus on the second term, labeled (2) above. We use the vector calculus relation

$$\nabla \cdot (f u_x \mathbf{u}) = u_x (\mathbf{u} \cdot \nabla f) + f \nabla \cdot (u_x \mathbf{u}) \quad (1.115)$$

The rightmost term vanishes because \mathbf{u} and \mathbf{x} are independent variables. For the same reason, we can take the ∇ operator outside of the integral.

$$(2) = \nabla \cdot \left(\int_{\mathbb{R}^3} m u_x \mathbf{u} f d^3 u \right) = \nabla \cdot \left(\int_{\mathbb{R}^3} m u_i u_j f d^3 u \right) \quad (1.116)$$

where we switch to Einstein's notations for the velocity coordinates. We now split the velocity into $u_i = v_i + w_i$ and develop the product as

$$(2) = \nabla \cdot \left(\int_{\mathbb{R}^3} m (v_i v_j + v_i w_j + v_j w_i + w_i w_j) f d^3 u \right) \quad (1.117)$$

The first of these 4 terms is the easiest to compute, as $v_i v_j$ is a constant in velocity space, so that

$$\nabla \cdot \left(v_i v_j \int_{\mathbb{R}^3} m f d^3 u \right) = \nabla \cdot (\rho v_i v_j) \quad (1.118)$$

The second and third terms both vanish, because the thermal velocity integrates to zero

$$\nabla \cdot \left(v_i \int_{\mathbb{R}^3} m w_j f d^3 u \right) = \nabla \cdot \left(v_j \int_{\mathbb{R}^3} m w_i f d^3 u \right) = 0 \quad (1.119)$$

The last term, on the other end, does not lead to any known macroscopic quantity. We therefore define the *pressure tensor* as the 3x3 matrix

$$P_{ij} = \int_{\mathbb{R}^3} m w_i w_j f d^3 u \quad (1.120)$$

so the final form of the second term of the first-order moment of the Boltzmann equation writes in tensor form

$$(2) = \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbb{P}) \quad (1.121)$$

Putting everything together, we get the macroscopic, fluid equation for momentum conservation

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbb{P}) = \rho \mathbf{a} \quad (1.122)$$

We cannot say much about this pressure tensor. Only by knowing the exact form of f can we compute this moment. We can only notice that its trace is familiar, because

$$\text{Tr } \mathbb{P} = \rho \sigma_x^2 + \rho \sigma_y^2 + \rho \sigma_z^2 = \rho \sigma_{3D}^2 \quad (1.123)$$

Energy conservation

Here we multiply the Boltzmann equation by $Q = \frac{1}{2}mu^2$. We will not derive the third conservation law, the conservation of total energy. This is left to the reader as exercise. We just summarize the main results. We have already defined the total energy

$$E = \frac{1}{2}\rho v^2 + e \quad (1.124)$$

where we define e as the *internal energy*. This is the kinetic energy associated with random, thermal motions.

$$e = \int \frac{1}{2}mw^2 f d^3u = \frac{1}{2}\text{Tr}(\mathbb{P}) \quad (1.125)$$

The derivation of the energy conservation equation follows the same methodology as the previous conservation laws. Familiar macroscopic quantities will emerge, like the pressure tensor. We will have to introduce a new macroscopic vector, defined as the *heat flux*

$$\mathbf{Q} = \int_{\mathbb{R}^3} m \frac{1}{2}w^2 \mathbf{w} f d^3u \quad (1.126)$$

The final form of the macroscopic energy conservation equation reads

$$\frac{\partial E}{\partial t} + \nabla \cdot (E \mathbf{v} + \mathbb{P} \cdot \mathbf{v} + \mathbf{Q}) = \rho \mathbf{a} \cdot \mathbf{u} \quad (1.127)$$

In summary, we have derived these three macroscopic conservation laws

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1.128)$$

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + \mathbb{P}) = \rho \mathbf{a} \quad (1.129)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (E \mathbf{v} + \mathbb{P} \cdot \mathbf{v} + \mathbf{Q}) = \rho \mathbf{a} \cdot \mathbf{v} \quad (1.130)$$

which are also known as the general fluid equations. These equations are valid for any underlying DF f . In practice, however, they are not really useful, as the form of \mathbb{P} and \mathbf{Q} remains unknown, unless we know the exact form of f , which requires to solve the Boltzmann equation. So we are back to square one. We will find explicit forms for these high-order moments in two limiting cases: the LTE regime and the Chapman-Enskog perturbation theory.

1.5.2 Euler equations for LTE conditions

When the fluid is at LTE, the DF takes a very specific form, namely the Maxwell-Boltzmann distribution

$$f_0(\mathbf{x}, \mathbf{u}, t) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} e^{-\frac{1}{2} \frac{m\mathbf{u}^2}{k_B T}} \quad (1.131)$$

This distribution is an even function of w . As a consequence, if multiply by an odd function of w , its integral will vanish. This leads to a great simplification because

$$P_{ij} = 0 \text{ for } i \neq j \text{ and } \mathbf{Q} = 0 \quad (1.132)$$

The pressure tensor is the only non-vanishing high-order moment, and it is diagonal. Because the Maxwell-Boltzmann distribution is also isotropic, we have

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \sigma^2 = \frac{k_B T}{m} \quad (1.133)$$

So the pressure tensor is now proportional to the identity matrix, with

$$\mathbb{P} = P\mathbb{I} \quad (1.134)$$

and $P = \rho\sigma^2$ is the *scalar pressure*. We obtain in this LTE limit the so-called Euler equations

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho\mathbf{v}) = 0 \quad (1.135)$$

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v} + P\mathbb{I}) = \rho\mathbf{a} \quad (1.136)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot ((E + P)\mathbf{v}) = \rho\mathbf{a} \cdot \mathbf{v} \quad (1.137)$$

Note that the momentum flux related to the scalar pressure can also be transformed in the *pressure gradient* using the relation

$$\nabla \cdot (P\mathbb{I}) = \nabla P \quad (1.138)$$

In the LTE case, we can also relate the internal energy to the gas pressure using $e = \frac{1}{2}\text{Tr } \mathbb{P}$, so that

$$e = \frac{3}{2}P \quad (1.139)$$

This last relation is also known as the ideal gas equation of state (noted in short the ideal gas EoS). Following up on the analogy with an ideal gas, we recall the relation connecting the velocity dispersion to the gas temperature, namely

$$P = \rho\sigma^2 = nk_B T \quad (1.140)$$

Interestingly enough, the Euler equations, presented here in their *Eulerian form*, are a closed system. Indeed, if one knows at time t , the mass, momentum and total energy densities, then, one can subtract to the total energy the fluid kinetic energy and deduce the fluid internal energy. Using the ideal gas equation of state, one then knows the scalar pressure, so one can compute the time derivatives to update the conservative variables $(\rho, \rho\mathbf{v}, E)$. There is therefore no need to use any microscopic properties, everything is specified at the macroscopic level. Let's stress again that this is valid only if LTE conditions are met, namely that all macroscopic variables scale lengths are much larger than the collision mean free path.

1.5.3 Euler equations in Lagrangian form

So far we have looked at the problem from an Eulerian perspective, meaning the observer is static and watches how the fluid move with respect to his/her reference frame. We will now look in the Lagrangian perspective, where the observer moves together with the fluid. The Lagrangian derivative, also known as comoving derivative, is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad (1.141)$$

The Euler equations can now be rewritten in Lagrangian form.

Mass conservation

Starting from the mass conservation equation in Eulerian form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0 \quad (1.142)$$

and using the now famous vector relation

$$\nabla \cdot (\rho \mathbf{v}) = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho \quad (1.143)$$

we find the mass conservation equation in Lagrangian form

$$\boxed{\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v}} \quad (1.144)$$

Momentum conservation

Using the Eulerian mass and momentum conservation laws, together with the definition of the Lagrangian derivative, it is possible to derive the Euler equation in Lagrangian form. We start with the momentum conservation equation for component v_x as

$$\frac{\partial}{\partial t}(\rho v_x) + \nabla \cdot (\rho v_x \mathbf{v}) + \frac{\partial P}{\partial x} = \rho a_x \quad (1.145)$$

By developing all time and space derivatives, one obtains the following form (left as an exercise)

$$\left(\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho \right) v_x + \rho \left(\frac{\partial v_x}{\partial t} + \mathbf{v} \cdot \nabla v_x \right) = \rho a_x - \frac{\partial P}{\partial x} \quad (1.146)$$

The first parenthesis on the left-hand side is nothing else but the mass conservation equation in Eulerian form, so it is equal to zero. The second parenthesis is the Lagrange derivative of v_x . The same applies for each component of the momentum, so we get in the end

$$\boxed{\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \rho \mathbf{a}} \quad (1.147)$$

Note that this equation can be obtained directly by applying Newton's second law, adding the pressure forces to the external acceleration as minus the pressure gradient. This is clearly the easiest way to remember it.

Energy conservation

A useful trick is to visualize a Lagrangian fluid element as a small volume of fluid containing a constant mass. If we label this volume V and its mass M , the fluid density is just

$$\rho = \frac{M}{V} \quad (1.148)$$

The conservation of mass can be written as

$$\frac{1}{M} \frac{DM}{Dt} = \frac{1}{V} \frac{DV}{Dt} + \frac{1}{\rho} \frac{D\rho}{Dt} = 0 \quad (1.149)$$

For a unit mass, V is called the *specific volume*, or the volume per unit mass. Using the mass conservation equation in Lagrangian form, we deduce

$$\frac{1}{V} \frac{DV}{Dt} = \nabla \cdot \mathbf{v} \quad (1.150)$$

which states that the rate of change of the volume is given by the divergence of the velocity field, a very useful interpretation.

Now we define the total internal energy (in units of [erg]) in our fluid element as

$$E = eV = M\epsilon \quad (1.151)$$

where e is the internal energy density defined earlier, or the internal energy per unit volume (in units of [erg cm⁻³]), and ϵ is the specific internal energy, or the internal energy per unit mass (in units of [erg g⁻¹]). We now use the first law of thermodynamics that states that $dE = -PdV$, or in Lagrangian form

$$\frac{DE}{Dt} = M \frac{D\epsilon}{Dt} = -P \frac{DV}{Dt} \quad (1.152)$$

Using the rate of change of the volume derived above, we finally get

$$\boxed{\rho \frac{D\epsilon}{Dt} = -P \nabla \cdot \mathbf{v}} \quad (1.153)$$

This is the energy equation in Lagrangian form. Note that we could have obtained the same result, starting with the energy equation in Eulerian form, then using both mass and momentum conservation, together with the definition of the Lagrange derivative, at the expense of relatively tedious calculations (left to the reader as exercise). The current approach, based on the first law of thermodynamics, is much easier to remember and strictly equivalent.

1.6 Chapman-Enskog theory

We have derived in the previous section the Euler equations, which are the fluid equations in the limiting case of LTE. The Chapman-Enskog theory will allow us to also derive a complete set of fluid equations, which are valid slightly outside LTE conditions. We will introduce fluid viscosity and heat conduction, two physical processes associated to the concept of dissipation. The Chapman-Enskog derivation is also a perturbative technique, used in many other domains of physics. The derivation we give here is a simplified version of the original Chapman-Enskog theory, a very rigorous perturbative approach of kinetic theory.

1.6.1 First-order Chapman-Enskog expansion of the DF

When we are not in LTE, the DF is not of Maxwell-Boltzmann form anymore. In the general case, we need to solve the full Boltzmann equation, which is, in most cases, untractable. We can however follow a perturbative approach, assuming that the fluid is reasonably close to LTE, and the general DF f is the Maxwell-Boltzmann DF f_0 plus a small additive perturbation $\Delta f \ll f_0$

$$f = f_0 + \Delta f \quad (1.154)$$

Starting with the full Boltzmann equation (without acceleration for sake of simplicity), we write

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = \mathcal{C}(f) \quad (1.155)$$

where the functional $\mathcal{C}(f)$ encodes the collision term as

$$\mathcal{C}(f) = \left(\frac{Df}{Dt} \right)_{\text{in}} - \left(\frac{Df}{Dt} \right)_{\text{out}} = \int_{\mathbb{R}^3} \int_{4\pi} (f'_1 f'_2 - f_1 f_2) \sigma v \, d\Omega \, d^3 u_2 \quad (1.156)$$

We now Taylor expand the collision term to first order, as

$$\mathcal{C}(f) \simeq \mathcal{C}(f_0) + \frac{\partial \mathcal{C}}{\partial f} \Delta f \quad (1.157)$$

By definition, we have $\mathcal{C}(f_0) = 0$. The partial derivative is however horribly complex and outside the scope of this course. We use here a very simple approximation, for which we assume that

$$\mathcal{C}(f) \simeq -\alpha(\mathbf{x}, t) \Delta f = \alpha(f_0 - f) \quad (1.158)$$

where α is just a function of position and time. We now determine α by computing the outgoing collision rate as

$$\int_{\mathbb{R}^3} \alpha f \, d^3 u_1 = \alpha n = n^2 \langle \sigma v \rangle = \frac{n}{\tau_{\text{coll}}} \quad (1.159)$$

so that

$$\alpha = \frac{1}{\tau_{\text{coll}}} \quad (1.160)$$

The Boltzmann equation can thus be written in the *collision time* approximation as

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = -\frac{\Delta f}{\tau_{\text{coll}}} \quad (1.161)$$

We now use the main trick of the Chapman-Enskog expansion, which is to match terms of the same order and neglect terms of higher-order. Injecting $f = f_0 + \Delta f$ in the previous equation, we neglect all Δf terms in the left-hand side. In the right-hand side, however, the Δf term is divided by τ_{coll} , which is also a very small quantity. Since the quotient of two small quantities

can be large, we have to keep the right-hand side as it is, and obtain the first-order expansion of the Boltzmann equation

$$\boxed{\frac{\partial f_0}{\partial t} + \mathbf{u} \cdot \nabla f_0 = -\frac{\Delta f}{\tau_{\text{coll}}}} \quad (1.162)$$

The problem is now for us to find a useful expression for Δf . Replacing the microscopic velocity by the sum of the bulk velocity and the thermal velocity $\mathbf{u} = \mathbf{v} + \mathbf{w}$, we can rewrite the previous equation as

$$\frac{\partial f_0}{\partial t} + \mathbf{v} \cdot \nabla f_0 + \mathbf{w} \cdot \nabla f_0 = -\frac{\Delta f}{\tau_{\text{coll}}} \quad (1.163)$$

$$\text{or } \frac{Df_0}{Dt} + \mathbf{w} \cdot \nabla f_0 = -\frac{\Delta f}{\tau_{\text{coll}}} \quad (1.164)$$

We now use the explicit form of the Maxwell-Boltzmann distribution

$$f_0 = \frac{\rho/m}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{1}{2} \frac{(\mathbf{u} - \mathbf{v})^2}{\sigma^2}\right) \quad (1.165)$$

but re-written in the following convenient form

$$\ln f_0 = \ln \rho - 3 \ln \sigma - \frac{1}{2} \frac{(\mathbf{u} - \mathbf{v})^2}{\sigma^2} + \text{constants} \quad (1.166)$$

Recall that the velocity dispersion is related to the temperature by $\sigma = \sqrt{\frac{k_B T}{m}}$. We can now take the logarithmic derivative of f_0 and obtain

$$\frac{f'_0}{f_0} = \frac{\rho'}{\rho} + \left(\frac{w^2}{\sigma^2} - 3\right) \frac{\sigma'}{\sigma} + \frac{\mathbf{w} \cdot \mathbf{v}'}{\sigma^2} \quad (1.167)$$

where f' denotes either the Lagrange derivative $\frac{Df}{Dt}$ or the gradient ∇f .

Using $P = \rho\sigma^2$ and $e = \frac{3}{2}\rho\sigma^2 = \rho\epsilon$, we can also rewrite the Euler equations in Lagrangian form in a more appropriate way:

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v} \quad (1.168)$$

$$\frac{D\mathbf{v}}{Dt} = -\frac{\nabla P}{\rho} = -\sigma^2 \frac{\nabla \rho}{\rho} - 2\sigma \nabla \sigma \quad (1.169)$$

$$\rho \frac{D\epsilon}{Dt} = -P \nabla \cdot \mathbf{v} \quad \text{or} \quad \frac{1}{\sigma} \frac{D\sigma}{Dt} = -\frac{1}{3} \nabla \cdot \mathbf{v} \quad (1.170)$$

Using these relations we get for the Lagrange time derivative

$$\frac{1}{f_0} \frac{Df_0}{Dt} = \left[-\nabla \cdot \mathbf{v} - \frac{1}{3} \nabla \cdot \mathbf{v} \left(\frac{w^2}{\sigma^2} - 3\right) + \mathbf{w} \cdot \left(-\frac{\nabla \rho}{\rho} - 2\frac{\nabla \sigma}{\sigma}\right) \right] \quad (1.171)$$

and for the second term featuring the gradient of f_0 , we get

$$\frac{1}{f_0} \mathbf{w} \cdot \nabla f_0 = \left[\mathbf{w} \cdot \frac{\nabla \rho}{\rho} + \mathbf{w} \cdot \left(\frac{w^2}{\sigma^2} - 3\right) \frac{\nabla \sigma}{\sigma} + \frac{\mathbf{w} \cdot (\nabla \mathbf{v}) \mathbf{w}}{\sigma^2} \right] \quad (1.172)$$

where we introduce in the third term the velocity gradient tensor

$$\mathbb{G} = \nabla \mathbf{v} = \frac{\partial v_i}{\partial x_j} \quad (1.173)$$

Finally, combining the time derivative and the gradient, we obtain for the perturbation an explicit form

$$\boxed{\frac{\Delta f}{f_0} = -\tau_{\text{coll}} \left[\left(\frac{w^2}{\sigma^2} - 5 \right) \frac{\mathbf{w} \cdot \nabla \sigma}{\sigma} - \frac{1}{3} (\nabla \cdot \mathbf{v}) \frac{w^2}{\sigma^2} + \frac{\mathbf{w} \cdot (\nabla \mathbf{v}) \mathbf{w}}{\sigma^2} \right]} \quad (1.174)$$

It is interesting to note that this expression is a polynomial of degree 3 in \mathbf{w} , and depends only on the gradient of σ (or T) and \mathbf{v} . More importantly for what follows, the temperature gradients are combined with a polynomial of degree 3, while the velocity gradients are combined with a polynomial of degree 2 in \mathbf{w} .

1.6.2 First-order expansion in the pressure tensor

Since \mathbb{P} and \mathbf{Q} are linear in f , we can write them as

$$P_{ij} = P \delta_{ij} + \Delta P_{ij} \quad (1.175)$$

$$Q_i = 0 + \Delta Q_i \quad (1.176)$$

where the LTE contribution is an isotropic scalar pressure for the pressure tensor and zero for the heat flux, so that their first-order correction can be computed as

$$\Delta P_{ij} = \int_{\mathbb{R}^3} m w_i w_j \Delta f d^3 w \quad (1.177)$$

$$\Delta Q_i = \int_{\mathbb{R}^3} m \frac{w^2}{2} w_i \Delta f d^3 w \quad (1.178)$$

At this point, it is more convenient to write Δf using explicit index summations

$$\frac{\Delta f}{f_0} = -\tau_{\text{coll}} \left[\left(\frac{w^2}{\sigma^2} - 5 \right) \sum_i \frac{w_i}{\sigma} \frac{\partial \sigma}{\partial x_i} + \sum_i \frac{w_i^2}{\sigma^2} \left(\frac{\partial v_i}{\partial x_i} - \frac{1}{3} (\nabla \cdot \mathbf{v}) \right) + \sum_i \sum_{j \neq i} \frac{w_i w_j}{\sigma^2} \frac{\partial v_i}{\partial x_j} \right] \quad (1.179)$$

We now compute the pressure tensor components, starting with the off-diagonal terms with $i \neq j$. Since f_0 is an even function of \mathbf{w} , all the terms in the integral that are odd in at least one component of \mathbf{w} are zero. This leaves us with only two surviving terms in the integral of Δf

$$\Delta P_{ij} = \int_{\mathbb{R}^3} m w_i w_j \Delta f d^3 w \quad (1.180)$$

$$= -\tau_{\text{coll}} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \int_{\mathbb{R}^3} m \frac{w_i^2 w_j^2}{\sigma^2} f_0 d^3 w \quad \text{for } i \neq j \quad (1.181)$$

Using $\int m w_i^2 w_j^2 f_0 d^3 w = \rho \sigma^4$, we get

$$\Delta P_{ij} = -\rho \tau_{\text{coll}} \sigma^2 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \text{for } i \neq j \quad (1.182)$$

The diagonal terms are more complicated, so we only focus on P_{xx} as an example that can be immediately generalized to P_{yy} and P_{zz}

$$\Delta P_{xx} = \int_{\mathbb{R}^3} m w_x^2 \Delta f d^3 w \quad (1.183)$$

Because now we are dealing with a second-order moment, we have more surviving terms in the integral

$$\begin{aligned}\Delta P_{xx} &= -\tau_{\text{coll}} \left(\frac{\partial v_x}{\partial x} - \frac{1}{3}(\nabla \cdot \mathbf{v}) \right) \int_{\mathbb{R}^3} m \frac{w_x^4}{\sigma^2} f_0 \, d^3w \\ &\quad -\tau_{\text{coll}} \left(\frac{\partial v_y}{\partial y} - \frac{1}{3}(\nabla \cdot \mathbf{v}) \right) \int_{\mathbb{R}^3} m \frac{w_x^2 w_y^2}{\sigma^2} f_0 \, d^3w \\ &\quad -\tau_{\text{coll}} \left(\frac{\partial v_z}{\partial z} - \frac{1}{3}(\nabla \cdot \mathbf{v}) \right) \int_{\mathbb{R}^3} m \frac{w_x^2 w_z^2}{\sigma^2} f_0 \, d^3w\end{aligned}\tag{1.184}$$

We easily compute the remaining moments with $\int w_x^4 G(w_x) \, dw_x = 3\sigma^4$, $\int w_x^2 G(w_x) \, dw_x = \sigma^2$, etc. After some simplifications, we get

$$\Delta P_{ii} = -\rho \tau_{\text{coll}} \sigma^2 \left(2 \frac{\partial v_i}{\partial x_i} - \frac{2}{3}(\nabla \cdot \mathbf{v}) \right)\tag{1.185}$$

Combining the results for diagonal and off-diagonal terms, we obtain the following compact tensor form for the pressure tensor

$$\mathbb{P} = P\mathbb{I} - \mu \left(\mathbb{G} + \mathbb{G}^T - \frac{2}{3}(\nabla \cdot \mathbf{v})\mathbb{I} \right)\tag{1.186}$$

where μ is the viscosity coefficient, that the Chapman-Enskog theory predicts to be

$$\mu = \rho \tau_{\text{coll}} \sigma^2\tag{1.187}$$

We will discuss how this coefficient depends on the macroscopic flow variables in the next section. A very important conclusion we would like to make here is the following: based on the first-order expansion we just performed, we are able to obtain self-consistently the additional term to the Euler equation traditionally called *viscosity*. The Chapman-Enskog theory provides a framework to derive the shape of the viscous tensor, based on the velocity gradient tensor, as well as the exact value of the viscosity coefficient.

1.6.3 First-order expansion in the heat flux

An analogous calculation can be done for the heat flux, which is defined as

$$\Delta Q_i = \int_{\mathbb{R}^3} m \frac{w^2}{2} w_i \Delta f \, d^3w\tag{1.188}$$

All velocity gradient terms vanish because we are dealing now with a third-order moment. We will derive only the heat flux for the x-component, with an obvious generalisation for the other two components. Using the previous equation on Δf , we see that the only non-vanishing term is

$$\Delta Q_x = -\tau_{\text{coll}} \frac{1}{\sigma} \frac{\partial \sigma}{\partial x} \int_{\mathbb{R}^3} m \frac{w^2}{2} w_x^2 \left(\frac{w^2}{\sigma^2} - 5 \right) f_0 \, d^3w\tag{1.189}$$

Developing $w^2 = w_x^2 + w_y^2 + w_z^2$ and using $\int w_x^6 G(w_x) \, dw_x = 15\sigma^6$, $\int w_x^4 G(w_x) \, dw_x = 3\sigma^4$, $\int w_x^2 G(w_x) \, dw_x = \sigma^2$, etc, we find

$$\int_{\mathbb{R}^3} m \frac{w^2}{2} w_x^2 \left(\frac{w^2}{\sigma^2} - 5 \right) f_0 \, d^3w = 10\rho\sigma^4\tag{1.190}$$

Finally, replacing the gradient of the velocity dispersion by the gradient of the temperature, using $\sigma^2 = \frac{k_B T}{m}$,

$$2 \frac{1}{\sigma} \frac{\partial \sigma}{\partial x} = \frac{1}{T} \frac{\partial T}{\partial x} \quad (1.191)$$

we obtain the final form of the heat flux vector

$$\boxed{\mathbf{Q} = -\kappa \nabla T} \quad (1.192)$$

where we introduce the heat conduction coefficient κ , which, according to the Chapman-Enskog theory, takes the value

$$\boxed{\kappa = 5\rho \tau_{\text{coll}} \sigma^4 \frac{1}{T}} \quad (1.193)$$

As for viscosity, the Chapman-Enskog theory allows us to derive self-consistently an additional energy flux to add to the energy equation. This new flux, encoding non-LTE effects, is proportional to the gradient of the temperature. This is a classical mechanism using routinely in thermal engineering models, known as Fourier's law. As a bonus, the Chapman-Enskog theory also gave us the exact value of the conduction coefficient κ .

1.6.4 Non-LTE modifications of the Euler equations

Now that we know the explicit form for the pressure tensor and the heat flux, we can close the general fluid equations using the following system of conservation laws

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1.194)$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + P\mathbb{I}) = \nabla \cdot (\mu \mathbb{S}) + \rho \mathbf{g} \quad (1.195)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P) \mathbf{v}] = \nabla \cdot (\mu \mathbb{S} \mathbf{v}) + \nabla \cdot (\kappa \nabla T) + \rho \mathbf{v} \cdot \mathbf{g} \quad (1.196)$$

where we introduced a new tensor called *the rate of strain tensor*, defined as

$$S_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}(\nabla \cdot \mathbf{v}) \delta_{ij} \quad (1.197)$$

Note that \mathbb{S} is symmetric and $\text{Tr}(\mathbb{S}) = 0$, so that purely rotating, expanding or compressing flows have no viscosity. Only shear flows produce enough strain to trigger viscosity. This is of course only true for pure Maxwell-Boltzmann gases. In case the particles have internal degrees of freedom, these conclusions do not apply and new terms will appear in these equations. Using standard vector and tensor calculus, one can also write the energy equation in Lagrangian form,

$$\boxed{\rho \frac{D\epsilon}{Dt} = -P \nabla \cdot \mathbf{v} + \mu (\mathbb{S} : \nabla \mathbf{v}) + \nabla \cdot (\kappa \nabla T)} \quad (1.198)$$

The operator $:$ stands for the contraction of the two tensors. Interestingly, this term is called *the dissipation function* and noted Φ . After some algebra, it is relatively straightforward to show that

$$\Phi = \mu (\mathbb{S} : \nabla \mathbf{v}) = 2\mu \sum_{i,j} \left[\frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{1}{3}(\nabla \cdot \mathbf{v}) \delta_{ij} \right]^2 \quad (1.199)$$

a quantity that is clearly always positive. This fact is particularly important because it demonstrates that these non-LTE terms are truly dissipative, or in other words, they always lead to an increase of the entropy, in agreement with Boltzmann H theorem.

1.6.5 Non-LTE effects as diffusion processes

We will now use a simple example to highlight the fact that heat conduction and viscosity can be seen as diffusion processes, diffusion of energy for the former and diffusion of momentum for the latter. We consider the case of a uniform density medium $\rho = \rho_0$ at rest $\mathbf{v} = 0$. In this case, the total energy is just the internal energy

$$E = e = \frac{3}{2}nk_B T = \frac{3}{2}\rho\sigma^2 \quad (1.200)$$

and the energy equation simplified to

$$\frac{\partial e}{\partial t} = \frac{3}{2}nk_B \frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T) \quad (1.201)$$

From this, obtain a simple PDE on the gas temperature with

$$\boxed{\frac{\partial T}{\partial t} = \nabla \cdot (\nu \nabla T)} \quad (1.202)$$

an equation also known as *the heat transfer equation*, with *diffusion coefficient*

$$\nu = \frac{\kappa}{\frac{3}{2}nk_B} \quad (1.203)$$

Diffusion processes are quite common in physics and can all be modelled using the heat transfer equation. A typical and very useful rule of thumb is that the diffusion coefficient, just based on dimensional argument, must be $\nu = VL$ where V is a typical velocity and L a typical length scale associated to the diffusion process. In our case, we have no choice than choosing for the velocity the thermal velocity dispersion σ and for the length scale the mean-free path λ_{coll} .

$$\nu = \sigma \lambda_{\text{coll}} = \sigma^2 \tau_{\text{coll}} \quad (1.204)$$

Injecting this in the relation connecting ν and κ , we immediately find

$$\kappa = \frac{3}{2}nk_B \sigma^2 \tau_{\text{coll}} = \frac{3}{2}\rho \sigma^4 \frac{1}{T} \tau_{\text{coll}} \quad (1.205)$$

which is almost exactly the same formula we obtained using Chapman-Enskog theory. This is a much easier derivation of the heat conduction coefficient.

We can use the same trick for the viscosity coefficient. We consider the simple case of a stationary flow in a pipe, where the x-velocity depends only on the y coordinate, and the y-velocity is zero. In this case, because the velocity divergence is zero, the density remains constant. Gravity balances the vertical pressure gradient, and the horizontal pressure gradient vanishes. The momentum conservation equation simplifies to

$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial}{\partial y} \left(\mu \frac{\partial v_x}{\partial y} \right) \longrightarrow \boxed{\frac{\partial v_x}{\partial t} = \frac{\partial}{\partial y} \left(\nu \frac{\partial v_x}{\partial y} \right)} \quad (1.206)$$

which is again the heat transfer equation with this time as diffusion coefficient $\nu = \mu/\rho$. Using the same trick as before, we can set $\nu = \sigma \lambda_{\text{coll}} = \sigma^2 \tau_{\text{coll}}$, and obtain directly the equation for the viscosity coefficient

$$\mu = \rho \sigma^2 \tau_{\text{coll}} \quad (1.207)$$

which is exactly the same equation than the one we derived using Chapman-Enskog theory, but recovered almost immediately.

1.6.6 Application to astrophysical gases

We can now compute the viscosity and conduction coefficients for typical cases found in astrophysics, namely neutral and molecular gases described by a constant cross section $\sigma_0 \simeq 10^{-15} \text{ cm}^2$ and ionized gases described by the Coulomb interaction. For the hard sphere case, one has

$$\frac{1}{\tau_{\text{coll}}} = n\sigma_0 \sqrt{\frac{k_B T}{m}} \quad (1.208)$$

so that the conduction coefficient writes

$$\kappa = nk_B \left(\frac{k_B T}{m} \right) \tau_{\text{coll}} = \frac{k_B}{\sigma_0} \sqrt{\frac{k_B T}{m}} \quad (1.209)$$

We see that it does not depend on gas density but only on the square root of the gas temperature. We also see that the conduction coefficient is smaller for larger particle masses. The viscosity coefficient, equivalently, can be computed in the hard sphere case as

$$\mu = \rho \left(\frac{k_B T}{m} \right) \tau_{\text{coll}} = \frac{m}{\sigma_0} \sqrt{\frac{k_B T}{m}} \quad (1.210)$$

Similarly, μ does not depend on gas density but on the square root of the temperature. On the other hand, the viscosity is now larger for larger particle masses.

We can also describe the case of a fully ionized plasma where we have positively charged protons and negatively charged free electrons. The other distinction between electrons and protons is their mass, which is $m_p = 1.66 \times 10^{-24} \text{ g}$ for protons and $m_e \simeq m_p/1836$ for electrons. When the system is at (or close to) LTE, the velocity dispersion will therefore be much larger for electrons than for protons

$$\frac{k_B T}{m_e} \gg \frac{k_B T}{m_p} \quad (1.211)$$

We have already seen that we can compute the electrons and protons cross sections as

$$\sigma_0 \simeq \frac{e^4}{(k_B T)^2} \ln \Lambda \quad (1.212)$$

The collision rate is proportional to the product of the cross-section with the thermal velocity dispersion. As a result, the collision rates will always be dominated by collisions with electrons, and one can in most cases neglect collisions with ions or other atoms and molecules.

On the other hand, when one combines these various quantities to compute the electrons and protons viscosity coefficients, we observe that the largest is the one for protons

$$\mu_p = \frac{m_p^3}{e^4 \ln \Lambda} \left(\frac{k_B T}{m_p} \right)^{5/2} \quad (1.213)$$

while the largest conduction coefficient is the one for electrons

$$\kappa_e = \frac{m_e^2}{e^4 \ln \Lambda} k_B \left(\frac{k_B T}{m_e} \right)^{5/2} \quad (1.214)$$

This is why in ionized gases we refer only to the electronic conduction and to the ionic viscosity.

1.7 Generalized kinetic theory

In this section, we will present a generalisation of kinetic theory to the case of relativistic and/or degenerate gases. This has important applications for the rest of the course, in particular to the internal structure of stellar objects and the stability of hydrostatic polytropes. This section also sets the scene for a more general formulation of kinetic theory based on general Hamiltonian dynamical systems, that will also prove useful to describe collisionless fluids.

1.7.1 Lagrangian and Hamiltonian mechanics

In Lagrangian mechanics, we use an arbitrary coordinate system $\mathbf{q}(t)$ to represent the particle positions, with which one can define an arbitrary Lagrangian as

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \quad (1.215)$$

Given this Lagrangian, the trajectories of the particles are found using *the least action principle*, minimizing the action given by

$$\mathcal{S} = \int_{t_0}^{t_1} \mathcal{L} dt \quad (1.216)$$

This minimization leads to the famous Euler-Lagrange equations defining the trajectories as

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad (1.217)$$

Defining the generalized momentum vector \mathbf{p} as

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}, t) \quad (1.218)$$

we can re-write the equation for the trajectories as

$$\dot{\mathbf{p}} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}, t) \quad (1.219)$$

In conclusion, for Lagrangian mechanics, we describe the trajectory using $(\mathbf{q}(t), \mathbf{p}(t))$, our new phase-space coordinates. We can invert the definition of the momentum to compute $\dot{\mathbf{q}}$, the Lagrangian value at time t and its spatial derivative. We can finally compute the new coordinate of the particle in phase-space at time $t + \Delta t$.

We can also use a strictly equivalent formulation, defining a new quantity called the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}, t) \equiv \mathbf{p} \cdot \dot{\mathbf{q}} - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \quad (1.220)$$

Differentiating H , and using the definition of \mathbf{p} and its time derivative, we get

$$dH = \dot{\mathbf{q}} \cdot d\mathbf{p} + \mathbf{p} \cdot d\dot{\mathbf{q}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot d\mathbf{q} - \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot d\dot{\mathbf{q}} - \frac{\partial \mathcal{L}}{\partial t} dt = \dot{\mathbf{q}} \cdot d\mathbf{p} - \dot{\mathbf{p}} \cdot d\mathbf{q} - \frac{\partial \mathcal{L}}{\partial t} dt \quad (1.221)$$

Identifying the partial derivatives of the Hamiltonian, we finally get Hamilton's equations that describe the trajectory in the framework of Hamiltonian mechanics.

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \quad (1.222)$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \quad (1.223)$$

This gives a more explicit formulation to obtain the trajectory in phase-space, as we do not need to invert the Lagrangian. Recall however that both formulations are strictly equivalent. To describe a physical system, we can either give the Lagrangian, and deduce the corresponding Hamiltonian, or set the Hamiltonian and compute the corresponding Lagrangian. Which formulation is the more convenient depends on the exact nature of each problem.

For a particular case, however, the Hamiltonian formulation appears more natural. This is for Lagrangian (or Hamiltonian) functions that do not depend explicitly on time.

$$\frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} = 0 \quad (1.224)$$

In this case, we have $H(\mathbf{q}, \mathbf{p})$, and using the chain rule, we get

$$\frac{dH}{dt} = \frac{\partial H}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial H}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} = -\dot{\mathbf{p}} \cdot \dot{\mathbf{q}} + \dot{\mathbf{q}} \cdot \dot{\mathbf{p}} = 0 \quad (1.225)$$

H is therefore a constant of motion along the particle's trajectory. This lead to a natural choice for the Hamiltonian, namely the total energy of the particle. In many cases, it can be written as

$$H(\mathbf{q}, \mathbf{p}) = K(\mathbf{p}) + V(\mathbf{q}) \quad (1.226)$$

where K is the kinetic energy and V is the potential energy for the external force. We will come back to this formulation later to describe relativistic particles.

1.7.2 Relativistic particles

Relativistic dynamics is based on the Lorentz transformation to change coordinates in space-time from the rest frame where the velocity of the particle is zero, to some other arbitrary frame with relative velocity \mathbf{v} . The particle 4-momentum, defined as

$$\mathbf{P} = (P^0, P^1, P^2, P^3) = (E/c, p_x, p_y, p_z) \quad (1.227)$$

is a frame-independent quantity, so that its norm in Minkovski space is conserved through a Lorentz transformation

$$-\frac{E^2}{c^2} + p_x^2 + p_y^2 + p_z^2 = -m^2 c^2 \quad (1.228)$$

where the right-hand side is the norm of the 4-momentum in the rest frame. This results in the definition of the energy of a relativistic particle as

$$E^2 = p^2 c^2 + m^2 c^4 \quad (1.229)$$

For elastic collisions, the particle rest mass will remain constant and one can define the Hamiltonian as

$$H = K(p) + V(q) \quad (1.230)$$

where K is the relativistic kinetic energy defined as

$$K = \sqrt{p^2 c^2 + m^2 c^4} - mc^2 \quad (1.231)$$

Hamilton's equation become in this case

$$\mathbf{v} = \dot{\mathbf{q}} = \frac{\partial K}{\partial \mathbf{p}} = \frac{\mathbf{p}c^2}{E} = \frac{\mathbf{p}}{\gamma m} \quad (1.232)$$

$$\dot{\mathbf{p}} = -\frac{\partial V}{\partial \mathbf{q}} \quad (1.233)$$

where we introduced the Lorentz boost factor, defined as

$$\gamma = \frac{E}{mc^2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (1.234)$$

We can recover two interesting limiting cases: the non-relativistic case for which $\gamma \simeq 1$, $v \ll c$ and $p \ll mc$, and the ultra-relativistic case for which $\gamma \gg 1$, $v \simeq c$ and $p \gg mc$. The kinetic energy for each case writes

$$\text{non-relativistic: } K = mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}} - mc^2 \simeq \frac{p^2}{2m} \quad (1.235)$$

$$\text{ultra-relativistic: } K = \sqrt{p^2 c^2 + m^2 c^4} - mc^2 \simeq pc \quad (1.236)$$

Hamilton's equation becomes quite simple in these two limiting cases, with

$$\text{non-relativistic: } \mathbf{v} = \frac{\partial K}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} \quad (1.237)$$

$$\text{ultra-relativistic: } \mathbf{v} = \frac{\partial K}{\partial \mathbf{p}} = c\mathbf{n} \quad (1.238)$$

where in the second case, $\mathbf{n} = \mathbf{p}/p$ is the unit vector pointing in the direction of the 3-momentum.

Before we address the question of generalizing Boltzmann equation, we need to discuss the issue of frame invariance. In physics, it is important to derive equations independent of the frame of reference. In a Galilean framework, this property is known as the Galilean invariance of the fluid equations. In the relativistic framework, this is not so easy, as time, space and 3-momentum are all affected differently by the Lorentz transformation.

The most important property of the Lorentz transformation we use here is the fact that a pure spatial volume element d^3x at fixed time will get modified by the Lorentz transform defined by velocity v and its corresponding Lorentz boost γ by

$$d^3x' = \gamma d^3x \quad (1.239)$$

The 3-momentum is also affected by the Lorentz transform so that

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \quad (1.240)$$

This is a frame invariant quantity. These two properties are left to the reader as an exercise. Interestingly, if we use the Lorentz transform to the comoving frame of the particle, we have $E = \gamma mc^2$ and the product of the two $d^3x d^3p = d^3x_0 d^3p_0$, where index 0 refers to the comoving frame. The corresponding phase-space element can therefore be considered also as frame-invariant, a property we will use immediately in the next section.

1.7.3 Generalized Boltzmann equation

Since our definition of phase-space has changed, we need to re-define the particle distribution function as

$$dN = f(\mathbf{q}, \mathbf{p}, t) d^3q d^3p \quad (1.241)$$

where $dV = d^3q d^3p$ is our new frame-invariant phase-space volume element and dN is the number of particle in this volume element. Since the particle number is frame-independent, we conclude that the DF is also a frame-invariant scalar. We will prove a generalized version of Liouville's theorem (which was introduced in the beginning of this chapter) using the following

coordinate transformation associated to the trajectory of all particles within an initial phase-space volume element $dV_0 = d^3q_0 d^3p_0$, over a small time step Δt .

$$\mathbf{q} = \mathbf{q}_0 + \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}_0, \mathbf{p}_0) \Delta t \quad (1.242)$$

$$\mathbf{p} = \mathbf{p}_0 - \frac{\partial H}{\partial \mathbf{q}}(\mathbf{q}_0, \mathbf{p}_0) \Delta t \quad (1.243)$$

The new volume element can be computed using $dV = \det(\mathbb{J}) dV_0$, where \mathbb{J} is the Jacobian matrix of the coordinate transformation, shown here for the one-dimensional case for simplicity

$$\mathbb{J} = \begin{bmatrix} 1 + \frac{\partial^2 H}{\partial p \partial q} \Delta t & \frac{\partial^2 H}{\partial p^2} \Delta t \\ -\frac{\partial^2 H}{\partial q^2} \Delta t & 1 - \frac{\partial^2 H}{\partial q \partial p} \Delta t \end{bmatrix} \quad (1.244)$$

We see that

$$\det(\mathbb{J}) = 1 + \left(\frac{\partial^2 H}{\partial q^2} \frac{\partial^2 H}{\partial p^2} - \left(\frac{\partial^2 H}{\partial q \partial p} \right)^2 \right) \Delta t^2 \quad (1.245)$$

so that the phase-space volume element remains constant in time. This property holds true for any Hamiltonian system. We also see that this is reminiscent of the Heisenberg uncertainty principle in quantum mechanics, as we have

$$d^3q d^3p = dV_0 = \text{constant} \quad (1.246)$$

while the uncertainty principle states that

$$\Delta q_x \Delta p_x \geq \frac{\hbar}{2} \simeq h \quad (1.247)$$

where Δq_x and Δp_x are the one-dimensional standard deviations in the position and in the momentum and \hbar is the reduced Planck's constant and h is the Planck constant. They don't have the same meaning at all. Liouville's theorem means that if a cloud of particles shrinks in position space, it has to expand in velocity space and vice-versa. Heisenberg uncertainty principle means that if one knows the position of a quantum particle with great accuracy, then one cannot know the momentum of the same particle very precisely, and vice-versa. On the other hand, the uncertainty principle introduces the interesting notion that there is a minimum phase-space volume element given by

$$\delta V_{\min} = h^3 \quad (1.248)$$

within which quantum effects will become important. We will come back to this point later.

Using Liouville's theorem and the conservation of the particle number (in absence of collisions), we conclude that f is also constant in time. Using the chain rule and Hamilton's equation, we obtain Boltzmann equation in the general case as

$$\frac{\partial f}{\partial t} + \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} = \frac{\partial f}{\partial t} + \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{Df}{Dt} \right)_{\text{coll}} \quad (1.249)$$

We finally get the compact form

$$\frac{\partial f}{\partial t} - \{H, f\} = \left(\frac{Df}{Dt} \right)_{\text{coll}} \quad (1.250)$$

where we define the *Poisson's brackets* of two scalars A and B as

$$\{A, B\} = \frac{\partial A}{\partial \mathbf{q}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{q}} \quad (1.251)$$

This form of Boltzmann's equation has many similarities with Schrödinger's equation in quantum mechanics, but this will take us too far away from our topic.

1.7.4 Generalized moments

We can define a frame-invariant first-order moment of the DF, using the relativistic 4-momentum $\mathbf{P} = (P^0, P^1, P^2, P^3)$ and integrating over the frame-invariant momentum-space volume element d^3p/E . We obtain the fluid *particle 4-flow* $\mathbf{N} = (N^0, N^1, N^2, N^3)$ for which

$$N^i = \int_{\mathbb{R}^3} P^i f \frac{1}{\gamma m} d^3p \quad (1.252)$$

Note that γm appears in the denominator instead of $E = \gamma mc^2$ to ensure proper units for the particle 4-flow vector. Since $P^0 = E/c = \gamma mc$, we get

$$N^0 = \int_{\mathbb{R}^3} cf d^3p = n(\mathbf{x}, t)c \quad (1.253)$$

where we recognize the particle number density. Using for the velocity $\mathbf{p} = \gamma m\mathbf{v}$, we also get

$$N^1 = N_x = \int_{\mathbb{R}^3} v_x f d^3p \quad (1.254)$$

which is the particle flux in the x-direction.

We can also define a second-order moment of the DF using the tensor product of the 4-momentum and integrating again over momentum space. We obtain the *energy-momentum tensor* \mathbb{T} as

$$\mathbb{T} = \int_{\mathbb{R}^3} \mathbf{P} \otimes \mathbf{P} f \frac{1}{\gamma m} d^3p \quad (1.255)$$

The first component of the tensor we compute is

$$T^{00} = \int_{\mathbb{R}^3} \frac{E^2}{c^2} f \frac{1}{\gamma m} d^3p = \int_{\mathbb{R}^3} Ef d^3p \quad (1.256)$$

where we used $E = \gamma mc^2$. We recognize here again the fluid energy density that can be decomposed into the fluid internal energy density and the fluid rest mass energy density as

$$T^{00} = \int_{\mathbb{R}^3} Kf d^3p + \rho(\mathbf{x}, t)c^2 = e(\mathbf{x}, t) + \rho(\mathbf{x}, t)c^2 \quad (1.257)$$

For sake of simplicity, we assume we are in the fluid *comoving frame*, so that there is no bulk kinetic energy. The space-time component of the tensor write (for x only)

$$T^{0x} = T^{x0} = \int_{\mathbb{R}^3} \frac{E}{c} p_x f \frac{1}{\gamma m} d^3p = c \int_{\mathbb{R}^3} p_x f d^3p \quad (1.258)$$

where we see the fluid x-momentum density. Finally, the space-space components are just

$$T^{ij} = P_{ij} = \int_{\mathbb{R}^3} p_i p_j f \frac{1}{\gamma m} d^3p = \int_{\mathbb{R}^3} v_i p_j f d^3p \quad (1.259)$$

for which we used $p_i = \gamma m v_i$. This is the momentum flux, also known as the pressure tensor (or stress tensor). We can always define a scalar pressure (even if the DF is not Maxwell-Boltzmann) as

$$P = \frac{1}{3} \text{Tr } \mathbb{P} = \frac{1}{3} \int_{\mathbb{R}^3} \mathbf{v} \cdot \mathbf{p} f d^3p = \frac{1}{3} \int_{\mathbb{R}^3} \frac{p^2 c^2}{E} f d^3p \quad (1.260)$$

We can study our two limiting case, namely non-relativistic (NR) or ultra-relativistic (UR). For the non-relativistic case, one has

$$\text{non-relativistic: } \mathbf{p} = m\mathbf{v}, \quad e = \int_{\mathbb{R}^3} \frac{p^2}{2m} f d^3p, \quad P = \frac{1}{3} \int_{\mathbb{R}^3} \frac{p^2}{m} f d^3p = \frac{2}{3}e \quad (1.261)$$

For the ultra-relativistic case, one has

$$\text{ultra-relativistic: } \mathbf{v} = c\mathbf{n}, \quad e = \int_{\mathbb{R}^3} pc f d^3p, \quad P = \frac{1}{3} \int_{\mathbb{R}^3} pc f d^3p = \frac{1}{3}e \quad (1.262)$$

In conclusion, if the DF is an even function of \mathbf{p} , the pressure tensor will be isotropic and proportional to the scalar pressure. In this case, the fluid is considered to be an ideal gas, with $\gamma = 5/3$ for the non-relativistic case (we knew that already) and $\gamma = 4/3$ for the ultra-relativistic case.

Using these new definitions, and in the absence of external forces, we can easily derive the moments of the Boltzmann equation. We obtain the *relativistic fluid equations*, which can be written in a compact form using the 4-position vector in space-time $\mathbf{X} = (ct, x, y, z)$. First, multiplying the Boltzmann equation by P^0 , and integrating over the momentum-space volume-element $d^3p/\gamma m$ we get the conservation of the particle number density as

$$\frac{\partial N^\alpha}{\partial X^\alpha} = 0 \quad (1.263)$$

Finally, multiplying it by $(P^0)^2$ and P^i , we get the conservation of energy and momentum as

$$\frac{\partial T^{\alpha\beta}}{\partial X^\alpha} = 0 \quad (1.264)$$

1.7.5 Quantum effects

As we have already discussed, quantum mechanics is governed by the Heisenberg uncertainty principle, which states that position q_x and momentum p_x cannot be known together with an accuracy better than

$$\Delta q_x \Delta p_x \simeq h \quad (1.265)$$

where h is the Planck constant. This introduces a natural *fixed* size for the quantum cell in phase-space

$$\delta V_{\min} = \Delta q^3 \Delta p^3 = h^3 \quad (1.266)$$

We define now the *occupation number* as the number of particles per phase-space quantum cell

$$\mathcal{N} = f(\mathbf{x}, \mathbf{p}, t) h^3 \quad (1.267)$$

The occupation number is a very important quantity in quantum mechanics, because it allows to define the two types of particles we are dealing with in nature, namely bosons and fermions. Fermions obey Pauli's exclusion principle: once a quantum cell is fully occupied by one fermion, no other fermion can join. Fermions are not social particles, they prefer to live alone. Bosons are the opposite, they are very social: not only do they accept to share their quantum cell with others, but they *prefer* to go into occupied cells.

Using kinetic theory, it is possible to encode this behavior using a correction factor to the collision integral. The outgoing collision rate, for example, must now be multiplied by the Bose enhancement factor $1 + \mathcal{N}'$ for bosons, or by the Fermi suppression factor $1 - \mathcal{N}'$ for fermions. Note that this quantum correction factors have to be applied to the phase-space volume element of the *products of the collision*, namely the prime particles. This gives us for bosons

$$\left(\frac{Df}{Dt}\right)_{\text{out}} = \int_{\mathbb{R}^3} \int_{4\pi} f_1 f_2 (1 + \mathcal{N}'_1)(1 + \mathcal{N}'_2) \sigma v d^3p_2 d\Omega \quad (1.268)$$

and for fermions

$$\left(\frac{Df}{Dt}\right)_{\text{out}} = \int_{\mathbb{R}^3} \int_{4\pi} f_1 f_2 (1 - \mathcal{N}'_1)(1 - \mathcal{N}'_2) \sigma v d^3p_2 d\Omega \quad (1.269)$$

The fact that the probability to get a collision from our current volume element to another volume element depends on the occupation number of this other volume element is a mind-blowing property of the quantum world. For photons, the Bose enhancement factor is often referred to as *induced emission* and is at the origin of the laser effect. Adding the incoming collisions, the collision integral now becomes

$$\left(\frac{Df}{Dt}\right)_{\text{coll}} = \int_{\mathbb{R}^3} \int_{\Omega} [f'_1 f'_2 (1 \pm \mathcal{N}_1)(1 \pm \mathcal{N}_2) - f_1 f_2 (1 \pm \mathcal{N}'_1)(1 \pm \mathcal{N}'_2)] \sigma v d^3 p_2 d\Omega \quad (1.270)$$

where the \pm symbol is referring either to bosons (+) or to fermions (-). We see that now detailed balance, as a sufficient condition for LTE, writes

$$\frac{f'_1 f'_2}{(1 \pm \mathcal{N}'_1)(1 \pm \mathcal{N}'_2)} = \frac{f_1 f_2}{(1 \pm \mathcal{N}_1)(1 \pm \mathcal{N}_2)} \quad (1.271)$$

In other words, recalling that $\mathcal{N} = fh^3$, the quantity

$$\ln \frac{\mathcal{N}}{1 \pm \mathcal{N}} \quad (1.272)$$

is now the new collision invariant. Since we have already 5 collision invariants, the 4-momentum components plus the rest mass, this new invariant has to be a linear combination of the others. We then get, using classical notations

$$\ln \frac{\mathcal{N}}{1 \pm \mathcal{N}} = \alpha + \beta E = \frac{\mu - E}{k_B T} \quad (1.273)$$

where we assume here that we are in the comoving frame of the fluid so that there is no dependance on the fluid velocity. The two remaining momentum-independent quantities are $k_B T$, the temperature and μ , called the *chemical potential*. Solving for \mathcal{N} results in the Fermi-Dirac distribution function

$$f = \frac{g}{h^3} \frac{1}{e^{(E-\mu)/k_B T} + 1} \quad \text{for fermions,} \quad (1.274)$$

and in the Bose-Einstein distribution function

$$f = \frac{g}{h^3} \frac{1}{e^{(E-\mu)/k_B T} - 1} \quad \text{for bosons.} \quad (1.275)$$

Note that we multiplied the DF by an extra factor g , which is the degeneracy of the state associated to the corresponding energy. For example, electrons can have two different spins ($s = 1/2$ or $s = -1/2$). In this case, they can occupy the same quantum cell, so that $g = 2$. Photons, on the other end, are bosons, so that the quantum cell can be filled up with no limitation, but they nevertheless have two spin values ($s = 1$ or $s = -1$), so that $g = 2$ for photons, too.

These new distribution functions are the equivalent of the Maxwell-Boltzmann one, but for non-classical particles. We defer the description of photons (our bosons in this course) to the chapter about radiation. We now describe in more details fermions, and how quantum effects can affect the fluid equation of state.

1.7.6 Degenerate gases

For a system of fermions at LTE, the distribution function is given by the Fermi-Dirac distribution, which, in the comoving frame of the fluid, is just

$$f = \frac{g}{h^3} \frac{1}{e^{(E-\mu)/k_B T} + 1} \quad (1.276)$$

where the relativistic energy is $E = \sqrt{p^2c^2 + m^2c^4} = K + mc^2$. Defining the Fermi energy as

$$K_F = \mu - mc^2 \quad (1.277)$$

one can rewrite the DF as

$$f = \frac{g}{h^3} \frac{1}{e^{(K-K_F)/k_B T} + 1} \quad (1.278)$$

We see that the Fermi energy marks the transition around $\mathcal{N} = 0.5$. For lower kinetic energies $K < K_F$, the quantum cells are almost completely occupied and $\mathcal{N} \simeq 1$, while at larger kinetic energies, $\mathcal{N} \ll 1$, and the quantum cells are almost completely free. We will consider three different regimes, the non-degenerate case, for which $\mathcal{N} \ll 1$ everywhere is phase-space, the partly degenerate case, for which $\mathcal{N} \simeq 1$ at low energies but $\mathcal{N} < 1$ at higher energies, and the fully degenerate case where \mathcal{N} takes only two values, either 1 at low energies and 0 at high energies.

Non-degenerate gases

We start with the non-degenerate case, which is defined by $\mathcal{N} \ll 1$ everywhere in phase-space. In particular, since the largest value for \mathcal{N} is obtained at $K = 0$, the condition for non-degenerate fluids can be written as $\exp^{-K_F/k_B T} \gg 1$. We can then neglect the +1 term in the Fermi-Dirac distribution and approximate f by

$$f \simeq \frac{g}{h^3} \exp^{\frac{K_F}{k_B T}} \exp^{-\frac{K}{k_B T}} \quad (1.279)$$

For *non-relativistic* gases, for which $K = p^2/2m$, we recognize the Maxwell-Boltzmann distribution function. Using the definition for the number density n , we have

$$n(\mathbf{q}, t) = \int_{\mathbb{R}^3} f d^3p \simeq \frac{g}{h^3} \exp^{\frac{K_F}{k_B T}} \int_0^{+\infty} \exp^{\frac{-p^2}{2mk_B T}} 4\pi p^2 dp = \frac{g}{h^3} \exp^{\frac{K_F}{k_B T}} (2\pi mk_B T)^{3/2} \quad (1.280)$$

From this last equation, we deduce the value of the Fermi energy and re-write the condition for non-degenerate gases as

$$\exp^{\frac{K_F}{k_B T}} = \frac{nh^3}{g(2\pi mk_B T)^{3/2}} < 1 \quad (1.281)$$

For *ultra-relativistic* gases, for which $K = pc$, we get

$$n(\mathbf{q}, t) = \int_{\mathbb{R}^3} f d^3p \simeq \frac{g}{h^3} \exp^{\frac{K_F}{k_B T}} \int_0^{+\infty} \exp^{\frac{-pc}{k_B T}} 4\pi p^2 dp = \frac{g}{h^3} \exp^{\frac{K_F}{k_B T}} 8\pi (k_B T/c)^3 \quad (1.282)$$

and the conditions for being non-degenerate translates into

$$\exp^{\frac{K_F}{k_B T}} = \frac{nh^3}{g8\pi (k_B T/c)^3} < 1 \quad (1.283)$$

We see that in both cases, if the gas is too dense or too cold, it will become degenerate, as \mathcal{N} will approach 1 dangerously.

Degenerate gases

In this case, we have no other choice but to use the Fermi-Dirac distribution. The Fermi energy can be determined using the number density as

$$n(\mathbf{q}, t) = \int_{\mathbb{R}^3} f d^3p = \frac{g}{h^3} \int_0^{+\infty} \frac{1}{\exp^{\frac{K_F - K}{k_B T}} + 1} 4\pi p^2 dp \quad (1.284)$$

In the last equation, we see that $k_B T$ controls how fast the distribution transitions at $K = K_F$ from $\mathcal{N} \simeq 1$ to $\mathcal{N} \simeq 0$. This leads us to define another regime for *ultra-degenerate gases*, for which

$$k_B T \ll K_F \quad (1.285)$$

In this case, we can approximate \mathcal{N} as the Heaviside function. Defining the Fermi momentum as $K_F = K(p_f) = \sqrt{p_f^2 c^2 + m^2 c^4} - mc^2$, we finally obtain

$$n = \frac{g}{h^3} \int_0^{p_F} 4\pi p^2 dp = \frac{4\pi g}{3h^3} p_F^3 \quad (1.286)$$

We can also compute easily the scalar pressure and the internal energy density from the previous section as

$$P = \frac{1}{3} \frac{g}{h^3} \int_0^{p_F} \frac{p^2 c^2}{E} 4\pi p^2 dp \quad \text{and} \quad e = \frac{g}{h^3} \int_0^{p_F} K 4\pi p^2 dp \quad (1.287)$$

We now consider the limiting case of an *ultra-degenerate and ultra-relativistic gas*, for which $E \simeq K \simeq pc$ and

$$P \simeq \frac{1}{3} \frac{g}{h^3} \int_0^{p_F} pc 4\pi p^2 dp = \frac{\pi c g}{3h^3} p_F^4 = \frac{\pi g c h}{3} \left(\frac{3n}{4\pi g} \right)^{4/3} = A \left(\frac{\rho}{m} \right)^{4/3} \quad (1.288)$$

This is a so-called polytropic equation of state, for which the pressure depends only on the gas density, and not on the temperature anymore. The constant A in this case is just

$$A = \frac{ch}{4} \left(\frac{3}{4\pi g} \right)^{1/3} \quad (1.289)$$

The *ultra-degenerate and non-relativistic case* leads to a different result, because we have now $E \simeq mc^2$ and $K \simeq p^2/2m$, so that

$$P \simeq \frac{1}{3} \frac{g}{h^3} \int_0^{p_F} \frac{p^2}{m} 4\pi p^2 dp = \frac{4\pi g}{15mh^3} p_F^5 = \frac{4\pi g h^2}{15m} \left(\frac{3n}{4\pi g} \right)^{5/3} = B \left(\frac{\rho}{m} \right)^{5/3} \quad (1.290)$$

which is a different polytrope with constant B given by

$$B = \frac{h^2}{5m} \left(\frac{3}{4\pi g} \right)^{2/3} \quad (1.291)$$

In conclusion, if the gas temperature $k_B T \ll mc^2$, we are in the non-relativistic regime. For densities lower than the critical density

$$n_{\text{crit}} = \frac{g}{h^3} (2\pi m k_B T)^{3/2} \quad (1.292)$$

the gas is non-degenerate, and we are in the Maxwell-Boltzmann regime and the pressure is given by $P = nk_B T = \frac{2}{3}e$. For densities larger than the critical density, however, we enter the regime of degenerate gases, for which $P \propto \rho^{5/3}$ independent on the temperature. If now we consider temperature $k_B T \gg mc^2$, we enter the ultra-relativistic regime. For densities lower than the critical density

$$n_{\text{crit}} = \frac{g}{h^3} 8\pi \left(\frac{k_B T}{c} \right)^3 \quad (1.293)$$

the gas pressure is $P = nk_B T = \frac{1}{3}e$. For larger densities, however, we enter the ultra-degenerate regime for which $P \propto \rho^{4/3}$.

Chapter 2

Astrophysical fluid dynamics

After deriving the Euler equations from first principles using kinetic theory, and the non-LTE source terms associated to viscosity and heat conduction, we are now in a position to use these equations to describe astrophysical fluid flows. In this chapter, we will address first the dynamics and the equilibrium of astrophysical systems, which can be either a galaxy, a star or a molecular cloud. These systems can be studied as a single object, using integral forms of the Euler equations. We can also describe their internal structure using various geometrical approximation, namely adopting spherical or cylindrical coordinate systems. We will describe in particular accretion flows within accretion disks and spherical Bondi stationary flows. The key question is: are those equilibrium or stationary solutions stable? This leads to the notion of waves, either propagating with a fixed amplitude (in other words stable) or propagating with an exponentially growing amplitude (in other words unstable). Among different types of waves, we will pay particular attention to shock waves, which play an important role in explosive astrophysical events such as supernovae. We will also describe important hydrodynamical instabilities at work in the interstellar medium or inside stars, leading to the theoretical description of astrophysical turbulence.

2.1 Euler equations in integral form

In order to describe global properties of entire systems, we will derive and use integral forms for the Euler equation. So far, we have the so-called Eulerian, conservative form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.1)$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + P\mathbb{I}) = \rho \mathbf{g} \quad (2.2)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P) \mathbf{v}] = \rho \mathbf{v} \cdot \mathbf{g} \quad (2.3)$$

where the gravitational acceleration is described by the vector $\mathbf{g} = -\nabla\phi$ and the gravitational potential ϕ is the solution of Poisson's equation

$$\Delta\phi = 4\pi G\rho \quad (2.4)$$

We have also derived the Euler equations in Lagrangian form, using the Lagrange time derivative, or the time derivative *following the fluid motions*.

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{v} \quad (2.5)$$

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla P + \rho\mathbf{g} \quad (2.6)$$

$$\rho\frac{D\epsilon}{Dt} = -P\nabla\cdot\mathbf{v} \quad (2.7)$$

In what follows, we will derive a new, very useful form of the Euler equations, called *integral form*.

2.1.1 Reynolds Transport Theorem

Before we embark into the derivation of these integral relations, we first need a little help from a well-know theorem for fluid dynamics, namely Reynolds transport theorem. We define a time-dependent scalar quantity I as the integral of an arbitrary fluid quantity $\alpha(\mathbf{x}, t)$.

$$I = \int_{V(t)} \alpha d^3x \quad (2.8)$$

where $V(t)$ is the Lagrangian volume of the system. This volume is not fixed, as it moves and deforms as the system evolves. We are interested in the time derivative of the integral quantity I .

The main trick of Reynolds transport theorem is to introduce a colour function $\psi(\mathbf{x})$, to indicate whether a fluid element \mathbf{x} is located inside or outside the volume $V(t)$:

$$\psi(\mathbf{x}) = 1 \quad \text{if } \mathbf{x} \in V(t) \quad (2.9)$$

$$\psi(\mathbf{x}) = 0 \quad \text{if } \mathbf{x} \notin V(t) \quad (2.10)$$

This colour function can be seen as painting particles initially inside the volume $V(t)$ in white and those initially outside in black. We can express the integral over the volume as an integral over the entire space.

$$I = \int_{V(t)} \alpha d^3x = \int_{\mathbb{R}^3} \alpha\psi d^3x \quad (2.11)$$

Since the integration bounds are now fixed, we can easily compute the time derivative of I as

$$\dot{I}(t) = \int_{\mathbb{R}^3} \left[\frac{\partial\alpha}{\partial t}\psi + \alpha\frac{\partial\psi}{\partial t} \right] d^3x = \int_{\mathbb{R}^3} \left[\psi\frac{\partial\alpha}{\partial t} - \alpha\mathbf{v}\cdot\nabla\psi \right] d^3x \quad (2.12)$$

where we used in the rightmost equation the property that the colour function does not change if you move with the fluid. In other words, its Lagrange time derivative is zero.

$$\frac{D\psi}{Dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla\psi = 0 \quad (2.13)$$

We now use the vector calculus relation $\nabla\cdot(\alpha\psi\mathbf{v}) = \psi\nabla\cdot(\alpha\mathbf{v}) + \alpha\mathbf{v}\cdot\nabla\psi$ and obtain

$$\dot{I}(t) = \int_{\mathbb{R}^3} \left(\psi \left[\frac{\partial\alpha}{\partial t} + \nabla\cdot(\alpha\mathbf{v}) \right] - \nabla\cdot(\alpha\psi\mathbf{v}) \right) d^3x \quad (2.14)$$

In order to work out the second term in the integral, we use the divergence theorem

$$\int_{V_0} \nabla\cdot\mathbf{v} d^3x = \int_{S_0} \mathbf{v}\cdot\mathbf{n} dS \quad (2.15)$$

applied to the vector $\alpha\psi\mathbf{v}$, using a fixed control volume much larger than $V(t)$. This gives us

$$\int_{V_0} \nabla \cdot (\alpha\psi\mathbf{v}) d^3x = \int_{S_0} \alpha\psi\mathbf{v} \cdot \mathbf{n} dS = 0 \quad (2.16)$$

because $\psi = 0$ on the boundary of this larger control volume since the boundary is entirely outside $V(t)$. We now take the limit $V_0 \rightarrow +\infty$ so that the integral over \mathbb{R}^3 also vanishes. We thus obtain the final result of Reynolds transport theorem.

$$\boxed{\frac{dI}{dt} = \int_{V(t)} \left[\frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha\mathbf{v}) \right] d^3x} \quad (2.17)$$

2.1.2 Integral conservation laws

Remember that we derived fluid or macroscopic conservation laws by integrating over velocity (or momentum) space the Boltzmann equation using 3 microscopic conservation laws. We now derive *system or integral* conservation laws by integrating the Euler equations over the volume of our system.

The total mass of the system is given by the sum of the mass of all its fluid elements

$$M = \int_{V(t)} \rho d^3x \quad (2.18)$$

Using the Reynolds transport theorem, we can now easily find the rate of change of the total mass as

$$\frac{dM}{dt} = \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) \right] d^3x = 0 \quad (2.19)$$

Indeed, the quantity between brackets is nothing else than the continuity equation, which is zero, hence $\dot{M} = 0$. The mass of the system defined by $V(t)$ is therefore a constant. The key point here is that the volume $V(t)$ is the Lagrangian volume that moves and deforms together with the system. This reasoning is different from taking the integral over a fixed volume V_0 independent of the system. In this case

$$M_0 = \int_{V_0} \rho d^3x \quad (2.20)$$

and

$$\dot{M}_0 = \int_{V_0} \frac{\partial \rho}{\partial t} d^3x = - \int_{V_0} \nabla \cdot (\rho\mathbf{v}) d^3x = - \int_{S_0} \rho\mathbf{v} \cdot \mathbf{n} dS \quad (2.21)$$

which expressed the total mass flux in/out of the volume.

We can also define the total momentum of the system as

$$\mathbf{P} = \int_{V(t)} \rho\mathbf{v} dV = M\mathbf{V} \quad (2.22)$$

where we exploit the fact that the mass is a constant and we introduced the system velocity \mathbf{V} . Taking the time derivative of the total momentum and applying Reynolds transport theorem, we get

$$M \frac{d\mathbf{V}}{dt} = \int_{V(t)} \left[\frac{\partial}{\partial t} (\rho\mathbf{v}) + \nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v}) \right] dV = \int_{V(t)} \rho\mathbf{g} dV - \int_{V(t)} \nabla \cdot (P\mathbb{I}) dV \quad (2.23)$$

Defining the mean system gravity as

$$M\mathbf{G} = \int_{V(t)} \rho \mathbf{g} dV \quad (2.24)$$

and using the divergence theorem applied to the isotropic pressure tensor

$$\int_{V(t)} \nabla \cdot (P\mathbb{I}) dV = \int_{S(t)} P \mathbf{n} dS \quad (2.25)$$

we get the system momentum conservation equation

$$\boxed{M \frac{d\mathbf{V}}{dt} = M\mathbf{G} - \int_{S(t)} P \mathbf{n} dS} \quad (2.26)$$

We see that we can describe the trajectory of the system as a whole very much like a single particle of mass M , velocity \mathbf{V} and gravity \mathbf{G} , with an additional *drag force* due to the pressure field applied to the surface of the system.

2.1.3 Specific variables

A special and very useful case of the Reynolds transport theorem can be derived, when α can be expressed as the product of the density and what is called a *specific variable*: $\alpha = \rho\beta$. If α represents a physical quantity *per unit volume*, β represents the same quantity *per unit mass*. For example, the specific internal energy ϵ , defined as $e = \rho\epsilon$, has units of [erg/g], while e , the internal energy, has units of [erg/cm³]. As usual, the integral form is written as

$$I(t) = \int_{V(t)} \rho\beta dV \quad (2.27)$$

Using Reynolds transport theorem, we write its time derivative as

$$\dot{I}(t) = \int_{V(t)} \left[\frac{\partial}{\partial t}(\rho\beta) + \nabla \cdot (\rho\beta\mathbf{v}) \right] dV \quad (2.28)$$

$$= \int_{V(t)} \left[\beta \frac{\partial \rho}{\partial t} + \rho \frac{\partial \beta}{\partial t} + \beta \nabla \cdot (\rho\mathbf{v}) + \rho\mathbf{v} \cdot \nabla \beta \right] dV \quad (2.29)$$

Using the continuity equation and the definition of the Lagrange time derivative, we obtain the fundamental relation

$$\boxed{\frac{d}{dt} \int_{V(t)} \rho\beta dV = \int_{V(t)} \rho \frac{D\beta}{Dt} dV} \quad (2.30)$$

We call this equation the second form of Reynolds transport theorem. We can for example define the system centre of mass coordinates as the vector \mathbf{X}

$$M\mathbf{X} = \int_{V(t)} \rho \mathbf{x} dV \quad (2.31)$$

Using the second form of Reynolds transport theorem, we get

$$M \frac{d}{dt} \mathbf{X} = \int_{V(t)} \rho \frac{D\mathbf{x}}{Dt} dV = \int_{V(t)} \rho \mathbf{v} dV = M\mathbf{V} \quad (2.32)$$

We see that the centre of mass velocity is equal to the system velocity defined earlier. The position of the centre of mass is therefore the coordinate of this virtual particle that we use to describe the system as a whole.

We can also define the *specific volume* as $v = 1/\rho$, the volume per unit mass. The total volume is trivially computed as

$$V(t) = \int_{V(t)} 1 \, dV = \int_{V(t)} \rho v \, dV \quad (2.33)$$

According to the second form of Reynolds transport theorem, the rate of change of the volume of the system is now simply

$$\frac{dV}{dt} = \int_{V(t)} \rho \frac{Dv}{Dt} \, dV = \int_{V(t)} \frac{1}{v} \frac{Dv}{Dt} \, dV \quad (2.34)$$

We can also use the first form of Reynolds transport theorem with $\alpha = 1$ and get immediately

$$\frac{dV}{dt} = \frac{d}{dt} \int_{V(t)} 1 \, dV = \int_{V(t)} \nabla \cdot \mathbf{v} \, dV \quad (2.35)$$

Identifying the terms under the integral, we deduce that the rate of change of the specific volume is given by

$$\boxed{\frac{1}{v} \frac{Dv}{Dt} = \nabla \cdot \mathbf{v}} \quad (2.36)$$

a relation that proved handy in the first chapter, when we applied the first law of thermodynamics to the Euler equations in Lagrangian form.

2.2 Virial Theorem

The virial theorem is probably the most famous theorem in theoretical astrophysics. The idea is to derive an expression for the time evolution of the scalar moment of inertia of the system of interest. The moment of inertia tensor is defined as

$$I_{ij}(t) = \int_{V(t)} \rho x_i x_j \, dV \quad (2.37)$$

Taking the trace of the tensor, we can define the scalar moment of inertia as

$$I(t) = \int_{V(t)} \rho |\mathbf{x}|^2 \, dV \quad \text{where} \quad |\mathbf{x}|^2 = x^2 + y^2 + z^2 \quad (2.38)$$

We can also compute the global expansion or contraction rate of the system using the time derivative of the scalar moment of inertia, using the second form of Reynolds theorem

$$\dot{I}(t) = \frac{dI}{dt} = \int_{V(t)} 2\rho \mathbf{x} \frac{D\mathbf{x}}{Dt} \, dV = 2 \int_{V(t)} \rho \mathbf{x} \cdot \mathbf{v} \, dV \quad (2.39)$$

where we used for the last step the definition of the Lagrange time derivative as

$$\frac{D\mathbf{x}}{Dt} = \frac{\partial \mathbf{x}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{x} = \mathbf{v} \quad (2.40)$$

Let's assume the system is contracting, which corresponds to $\dot{I} < 0$. The second-order time derivative will tell us whether the system will keep on contracting at a faster and faster rate

$\ddot{I} < 0$, or if it will eventually stop contracting and even expand later $\ddot{I} > 0$. In the former case, we say that the system is *collapsing*. Using again Reynolds theorem, we have

$$\frac{1}{2}\ddot{I}(t) = \frac{1}{2}\frac{d^2I}{dt^2} = \int_{V(t)} \left[\rho v^2 + \rho \mathbf{x} \cdot \frac{D\mathbf{v}}{Dt} \right] dV \quad (2.41)$$

We can simplify this by using the Euler equation in Lagrangian form $\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \rho \mathbf{g}$, where g is the gravitational acceleration

$$\frac{1}{2}\ddot{I}(t) = \int_{V(t)} \rho v^2 dV + \int_{V(t)} \rho \mathbf{x} \cdot \mathbf{g} dV - \int_{V(t)} \mathbf{x} \cdot \nabla P dV \quad (2.42)$$

$$(1) \qquad (2) \qquad (3) \qquad (2.43)$$

The first term is exactly twice the kinetic energy of the system

$$(1) = 2K = \int_{V(t)} \rho v^2 dV \quad (2.44)$$

The second term is called the *virial* of the system. It comes from Latin *vis* that means force.

$$(2) = V \equiv \int_{V(t)} \rho \mathbf{x} \cdot \mathbf{g} dV \quad (2.45)$$

Note that this is in general not equal to the gravitational potential of the system, as the gravity can also come from an external objects. The third term is related to the scalar thermal pressure. Using the vector relation $\nabla \cdot (P\mathbf{x}) = \mathbf{x} \cdot \nabla P + P\nabla \cdot \mathbf{x}$, it can be decomposed into

$$(3) = - \int_{V(t)} \mathbf{x} \cdot \nabla P dV = - \int_{S(t)} P \mathbf{x} \cdot \mathbf{n} dS + \int_{V(t)} 3P dV = T - S \quad (2.46)$$

where we used the trick $\nabla \cdot \mathbf{x} = 3$. We can decompose this term into two contributions: first, the thermal contribution written as

$$T \equiv \int_{V(t)} 3P dV \quad (2.47)$$

and the surface term

$$S \equiv \int_{S(t)} P \mathbf{x} \cdot \mathbf{n} dS \quad (2.48)$$

This gives us the final form of the virial theorem as $\frac{1}{2}\ddot{I} = 2K + V + T - S$, or

$$\boxed{\frac{1}{2}\ddot{I}(t) = \int_{V(t)} \rho v^2 dV + \int_{V(t)} \rho \mathbf{x} \cdot \mathbf{g} dV + 3 \int_{V(t)} P dV - \int_{S(t)} P \mathbf{x} \cdot \mathbf{n} dS} \quad (2.49)$$

For a non-relativistic Maxwell-Boltzmann gas, we have $e = \frac{3}{2}P$, so that the pressure can be replaced by the internal energy using $2E = T$ and the kinetic energy is now the sum of the bulk kinetic energy and the thermal kinetic energy we can write the virial theorem in a compact form, quite easy to remember,

$$\frac{1}{2}\ddot{I} = 2(K + E) + V - S \quad (2.50)$$

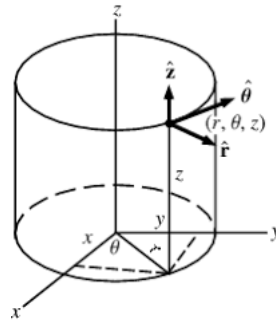
This is a powerful theorem that directly gives us information about the stability of a system. When $\ddot{I} < 0$, the system is collapsing; when $\ddot{I} > 0$ it is expanding. When $\ddot{I} = 0$, the terms balance each other and the system is in equilibrium.

Analyzing each term, we see that since the kinetic energy (bulk + internal) is always positive, it will always drive the system towards expansion. The virial is often negative, since \mathbf{g} points towards the main source of gravity, which, for an isolated system, is usually the center of mass. But V can also be positive, if tidal forces from an external object are also present. The surface term comes with a negative sign, driving the system towards contraction. This term corresponds to the pressure of the external environment onto the boundary of the system. If the external pressure is small compared to the internal pressure, which is usually the case for stars, we can ignore the surface term and set $S = 0$.

2.3 Euler equations in different coordinate systems

Often the system exhibits some form of symmetry. Many astrophysical objects are spheres or disks. To describe these, it is more convenient to express the Euler equations in cylindrical or spherical coordinates.

2.3.1 Euler equations in cylindrical coordinates



The relation between the cartesian coordinates (x, y, z) and the corresponding cylindrical coordinates (r, θ, z) is

$$x = r \cos \theta \quad (2.51)$$

$$y = r \sin \theta \quad (2.52)$$

$$z = z \quad (2.53)$$

with $r = \sqrt{x^2 + y^2}$. The Jacobian matrix of this coordinate transform is

$$\mathbb{J} = \begin{bmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.54)$$

We deduce that the new volume element writes

$$dx dy dz = |\det(\mathbb{J})| dr d\theta dz = r dr d\theta dz \quad (2.55)$$

We can find the associated orthonormal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$, so that

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z \quad (2.56)$$

using also the Jacobian matrix and renormalizing each column

$$\mathbf{e}_r \propto \frac{\partial \mathbf{r}}{\partial r}, \quad \mathbf{e}_\theta \propto \frac{\partial \mathbf{r}}{\partial \theta}, \quad \mathbf{e}_z \propto \frac{\partial \mathbf{r}}{\partial z} \quad (2.57)$$

We find the relation between the new basis and the old one as

$$\mathbf{e}_r = \cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y \quad (2.58)$$

$$\mathbf{e}_\theta = -\sin \theta \mathbf{e}_x + \cos \theta \mathbf{e}_y \quad (2.59)$$

$$\mathbf{e}_z = \mathbf{e}_z \quad (2.60)$$

We can also compute the derivatives of the new basis elements with respect to the new variables. The only non-zero derivatives are

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \quad (2.61)$$

We can now compute the differential form $d\mathbf{r}$ which represents a small variation of the position vector \mathbf{r} . Using the chain rule, we have

$$d\mathbf{r} = dr \mathbf{e}_r + r d\mathbf{e}_r + dz \mathbf{e}_z = dr \mathbf{e}_r + r d\theta \mathbf{e}_\theta + dz \mathbf{e}_z \quad (2.62)$$

The frame invariant definition of the gradient of a scalar function $f(r, \theta, z)$ is

$$df = \nabla f \cdot d\mathbf{r} \quad (2.63)$$

We can also compute it using the chain rule as

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial z} dz \quad (2.64)$$

Identifying the terms in the two relations, we get for the gradient vector of f in cylindrical coordinates

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)^T \quad (2.65)$$

or in vector form

$$\nabla f = \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \mathbf{e}_z \frac{\partial f}{\partial z} \quad (2.66)$$

The divergence of a vector can be computed using the frame invariant notation $\nabla \cdot \mathbf{v}$, taking literally the scalar product of the gradient operator and the vector $\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z$.

$$\nabla \cdot \mathbf{v} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot (v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z) \quad (2.67)$$

The only non-trivial terms are

$$\frac{\partial}{\partial \theta} (v_r \mathbf{e}_r) = \frac{\partial v_r}{\partial \theta} \mathbf{e}_r + v_r \mathbf{e}_\theta \quad (2.68)$$

$$\frac{\partial}{\partial \theta} (v_\theta \mathbf{e}_\theta) = \frac{\partial v_\theta}{\partial \theta} \mathbf{e}_\theta - v_\theta \mathbf{e}_r \quad (2.69)$$

Using the fact that we have an orthonormal basis, we finally get

$$\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \quad (2.70)$$

Using the same trick, we can also compute the Lagrange derivative of a scalar f in a frame invariant way as usual

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f \quad (2.71)$$

and take the scalar product between \mathbf{v} and the gradient operator in cylindrical coordinate to obtain

$$\boxed{\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r} \frac{\partial f}{\partial \theta} + v_z \frac{\partial f}{\partial z}} \quad (2.72)$$

The Lagrange derivative of the velocity field (or any vector field for that matter) is written in a coordinate system invariant as

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \quad (2.73)$$

We have already written the operator $\mathbf{v} \cdot \nabla$ in cylindrical coordinates,

$$(\mathbf{v} \cdot \nabla) = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \quad (2.74)$$

We apply it now to a vector field

$$(\mathbf{v} \cdot \nabla) (v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z) \quad (2.75)$$

which gives us the Lagrange derivative of the coordinates that remains parallel to the basis vector, but also of the basis vectors \mathbf{e}_r and \mathbf{e}_θ , which project to the other dimension

$$\boxed{\frac{D\mathbf{v}}{Dt} = \frac{Dv_r}{Dt} \mathbf{e}_r + \frac{Dv_\theta}{Dt} \mathbf{e}_\theta + \frac{Dv_z}{Dt} \mathbf{e}_z + \frac{v_\theta v_r}{r} \mathbf{e}_\theta - \frac{v_\theta^2}{r} \mathbf{e}_r} \quad (2.76)$$

Using the frame invariant, Lagrangian form of the momentum conservation equation

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla P - \nabla \phi \quad (2.77)$$

where we write the gravitational acceleration as $\mathbf{g} = -\nabla \phi$, we can now write the Euler equations in cylindrical coordinates

$$\frac{Dv_r}{Dt} = \frac{v_\theta^2}{r} - \frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{\partial \phi}{\partial r} \quad (2.78)$$

$$\frac{Dv_\theta}{Dt} = -\frac{v_r v_\theta}{r} - \frac{1}{\rho} \frac{1}{r} \frac{\partial P}{\partial \theta} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad (2.79)$$

$$\frac{Dv_z}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - \frac{\partial \phi}{\partial z} \quad (2.80)$$

where the geometrical terms have been placed on the right-hand side, together with the pressure gradient and the gravitational acceleration, and are called the *centrifugal acceleration*. The mass and energy conservation equations are usually expressed in conservative form using the expression of the divergence operator in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.81)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P) \mathbf{v}] = \rho \mathbf{g} \cdot \mathbf{x} \quad (2.82)$$

We can also express the stress tensor in cylindrical coordinates using the matrix, components form as

$$\mathbb{S} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix} \quad (2.83)$$

but also using a *tensorial basis* as

$$\mathbb{S} = \sum_i \sum_j S_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad (2.84)$$

Index i stands for line number i of the matrix, while index j stands for column number j . The velocity gradient tensor writes in Cartesian coordinates

$$\mathbb{G} = \sum_i \sum_j \frac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{v} \otimes \nabla \quad (2.85)$$

where the last form is conveniently independent of the coordinate system. The ∇ operator stands on the right of the tensor product \otimes , because the column number indicates which variables the derivative is taken from. The transpose of the gradient tensor is just

$$\mathbb{G}^T = \nabla \otimes \mathbf{v} \quad (2.86)$$

In cylindrical coordinates, we have

$$\mathbb{G} = (v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z) \otimes \left(\frac{\partial(\cdot)}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial(\cdot)}{\partial \theta} \mathbf{e}_\theta + \frac{\partial(\cdot)}{\partial z} \mathbf{e}_z \right) \quad (2.87)$$

Here again, we have to take the θ -derivative of the product $v_r \mathbf{e}_r$ as

$$v_r \mathbf{e}_r \otimes \frac{1}{r} \frac{\partial(\cdot)}{\partial \theta} \mathbf{e}_\theta = \frac{1}{r} \frac{\partial(v_r \mathbf{e}_r)}{\partial \theta} \otimes \mathbf{e}_\theta = \frac{1}{r} \frac{\partial v_r}{\partial \theta} \mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{v_r}{r} \mathbf{e}_\theta \otimes \mathbf{e}_\theta \quad (2.88)$$

and the θ -derivative of the product $v_\theta \mathbf{e}_\theta$ as

$$v_\theta \mathbf{e}_\theta \otimes \frac{1}{r} \frac{\partial(\cdot)}{\partial \theta} \mathbf{e}_\theta = \frac{1}{r} \frac{\partial(v_\theta \mathbf{e}_\theta)}{\partial \theta} \otimes \mathbf{e}_\theta = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \mathbf{e}_\theta \otimes \mathbf{e}_\theta - \frac{v_\theta}{r} \mathbf{e}_r \otimes \mathbf{e}_\theta \quad (2.89)$$

so that the velocity gradient tensor writes

$$\mathbb{G} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix} \quad (2.90)$$

Finally, the stress tensor, which is given by the coordinate-independent form,

$$\mathbb{S} = \mathbb{G} + \mathbb{G}^T - \frac{2}{3} (\nabla \cdot \mathbf{v}) \mathbb{I} \quad (2.91)$$

can be written in cylindrical coordinates as

$$\mathbb{S} = \begin{bmatrix} 2 \frac{\partial v_r}{\partial r} - \frac{2}{3} (\nabla \cdot \mathbf{v}) & \frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & 2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) - \frac{2}{3} (\nabla \cdot \mathbf{v}) & \frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \\ \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} & \frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} & 2 \frac{\partial v_z}{\partial z} - \frac{2}{3} (\nabla \cdot \mathbf{v}) \end{bmatrix} \quad (2.92)$$

To compute the viscous stress, we need to take the divergence of the stress tensor using the frame invariant form $\nabla \cdot \mathbb{S}$, which writes, using the ∇ operator and the tensorial basis as

$$\nabla \cdot \mathbb{S} = \left[\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right] \cdot \begin{bmatrix} S_{rr} \mathbf{e}_r \otimes \mathbf{e}_r + S_{r\theta} \mathbf{e}_r \otimes \mathbf{e}_\theta + S_{rz} \mathbf{e}_r \otimes \mathbf{e}_z \\ + S_{\theta r} \mathbf{e}_\theta \otimes \mathbf{e}_r + S_{\theta\theta} \mathbf{e}_\theta \otimes \mathbf{e}_\theta + S_{\theta z} \mathbf{e}_\theta \otimes \mathbf{e}_z \\ + S_{zr} \mathbf{e}_z \otimes \mathbf{e}_r + S_{z\theta} \mathbf{e}_z \otimes \mathbf{e}_\theta + S_{zz} \mathbf{e}_z \otimes \mathbf{e}_z \end{bmatrix} \quad (2.93)$$

For this, we need to take the derivative of the product of three terms such as, for example

$$\frac{1}{r} \frac{\partial}{\partial \theta} (S_{\theta\theta} \mathbf{e}_\theta \otimes \mathbf{e}_\theta) = \frac{1}{r} \frac{\partial S_{\theta\theta}}{\partial \theta} \mathbf{e}_\theta \otimes \mathbf{e}_\theta - \frac{S_{\theta\theta}}{r} \mathbf{e}_r \otimes \mathbf{e}_\theta - \frac{S_{\theta\theta}}{r} \mathbf{e}_\theta \otimes \mathbf{e}_r \quad (2.94)$$

and then take the scalar product with \mathbf{e}_θ using

$$\mathbf{e}_\theta \cdot (\mathbf{e}_\theta \otimes \mathbf{e}_\theta) = (\mathbf{e}_\theta \cdot \mathbf{e}_\theta) \mathbf{e}_\theta = \mathbf{e}_\theta \quad (2.95)$$

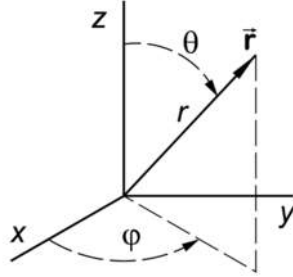
$$\mathbf{e}_\theta \cdot (\mathbf{e}_r \otimes \mathbf{e}_\theta) = (\mathbf{e}_\theta \cdot \mathbf{e}_r) \mathbf{e}_\theta = 0 \quad (2.96)$$

$$\mathbf{e}_\theta \cdot (\mathbf{e}_\theta \otimes \mathbf{e}_r) = (\mathbf{e}_\theta \cdot \mathbf{e}_\theta) \mathbf{e}_r = \mathbf{e}_r \quad (2.97)$$

We finally obtain the divergence of the tensor in cylindrical coordinates as

$$\nabla \cdot \mathbb{S} = \begin{bmatrix} \frac{\partial S_{rr}}{\partial r} + \frac{\partial S_{zr}}{\partial z} + \frac{1}{r} \frac{\partial S_{\theta r}}{\partial \theta} + \frac{S_{rr}}{r} - \frac{S_{\theta\theta}}{r} \\ \frac{\partial S_{r\theta}}{\partial r} + \frac{\partial S_{z\theta}}{\partial z} + \frac{1}{r} \frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{S_{r\theta}}{r} + \frac{S_{\theta r}}{r} \\ \frac{\partial S_{rz}}{\partial r} + \frac{\partial S_{zz}}{\partial z} + \frac{1}{r} \frac{\partial S_{\theta z}}{\partial \theta} \end{bmatrix} \quad (2.98)$$

2.3.2 Euler equations in spherical coordinates



The spherical coordinate system is defined by the transform between (x, y, z) and (r, θ, ϕ)

$$x = r \sin \theta \cos \phi \quad (2.99)$$

$$y = r \sin \theta \sin \phi \quad (2.100)$$

$$z = r \cos \theta \quad (2.101)$$

The Jacobian matrix is easily written as

$$\mathbb{J} = \begin{bmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{bmatrix} \quad (2.102)$$

Its determinant allows us to compute the new volume element

$$dx dy dz = r^2 \sin \theta dr d\theta d\phi \quad (2.103)$$

and renormalizing its columns gives us the new orthonormal basis vectors as

$$\mathbf{e}_r = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T \quad (2.104)$$

$$\mathbf{e}_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)^T \quad (2.105)$$

$$\mathbf{e}_\phi = (-\sin \phi, \cos \phi, 0)^T \quad (2.106)$$

We need to know how the derivatives of the basis vectors project in the new basis. The only non-zero terms are

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \quad \text{and} \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \quad (2.107)$$

$$\frac{\partial \mathbf{e}_r}{\partial \phi} = \sin \theta \mathbf{e}_\phi \quad \text{and} \quad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos \theta \mathbf{e}_\phi \quad \text{and} \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin \theta \mathbf{e}_r - \cos \theta \mathbf{e}_\theta \quad (2.108)$$

With all this information, we let the reader derive, as an exercise, all terms of the Euler equation in spherical coordinates.

2.4 Spherical systems in equilibrium

In most cases, we are interested in finding solutions for which the system is in equilibrium. Most things we observe in nature are in some form of equilibrium. Systems that are out of equilibrium evolve fast and are more difficult to see. In this section, we are interested into *static equilibrium states*, for which all time derivatives vanish $\frac{\partial}{\partial t}(\text{variables}) = 0$ and the velocity is also zero everywhere. We will see later examples of *stationary solutions*, for which all time derivatives vanish but the velocity field does not, like for example rotating or accretion flows.

2.4.1 Uniform spheres

One of the simplest systems imaginable is a spherical gas cloud of radius R with a uniform density, collapsing under its own gravity and possibly finding an equilibrium state. To find such equilibrium conditions, we write the virial theorem as

$$\frac{1}{2} \ddot{I} = 2K + T + V - S \quad (2.109)$$

Assuming static equilibrium for our system ($v = 0$)

$$K = \int_V \frac{1}{2} \rho v^2 dV = 0 \quad (2.110)$$

The thermal term can be also simplified as

$$T = \int_V 3P dV = 3P \left(\frac{4\pi}{3} R^3 \right) \quad (2.111)$$

and the surface term is simply

$$S = \int_S P_{\text{ext}} (R \mathbf{e}_r \cdot \mathbf{e}_r) dS = P_{\text{ext}} (4\pi R^3) \quad (2.112)$$

where we use the fact that the outward pointing normal vector $\mathbf{n} = \mathbf{e}_r$ and we define P_{ext} as the external pressure acting on the surface of the sphere.

Gravity of a uniform sphere

To find the virial V , we first have to compute the gravitational acceleration g inside the uniform sphere. The gravitational potential obeys the Poisson equation

$$\Delta \phi = 4\pi G \rho \quad \text{with} \quad \mathbf{g} = -\nabla \phi \quad (2.113)$$

We consider a sphere of radius $r < R$. Integrating over its volume, we get on one hand

$$\int_V \Delta\phi \, dV = 4\pi G \int_V \rho \, dV = 4\pi G M(< r) \quad (2.114)$$

where $M(< r)$ is the *enclosed mass*. For a uniform density sphere, the enclosed mass is just

$$M(< r) = \int_0^r \rho 4\pi r^2 \, dr = \rho \frac{4\pi}{3} r^3 \quad (2.115)$$

On the other hand, using the vector calculus relation $\Delta\phi = \nabla \cdot (\nabla\phi) = -\nabla \cdot \mathbf{g}$, we apply the divergence theorem on the same sphere of radius $r < R$

$$\int_V \Delta\phi \, dV = - \int_V \nabla \cdot \mathbf{g} \, dV = - \int_S \mathbf{g} \cdot \mathbf{n} \, dS = -g_r 4\pi r^2 \quad (2.116)$$

Combining these equations gives us the gravitational acceleration

$$\boxed{g_r = -\frac{GM(< r)}{r^2} = -\frac{4\pi}{3} G\rho r} \quad (2.117)$$

The virial of the uniform sphere now becomes

$$V = \int_V \rho \mathbf{g} \cdot \mathbf{x} \, dV = -\rho \int_0^R g_r r 4\pi r^2 \, dr = -\frac{4\pi}{3} G\rho^2 4\pi \frac{R^5}{5} \quad (2.118)$$

Virial equilibrium condition

This results into the equilibrium condition

$$\frac{1}{2} \ddot{I} = 3P \frac{4}{3} \pi R^3 - \frac{4\pi}{3} G\rho^2 4\pi \frac{R^5}{5} - P_{\text{ext}} 4\pi R^3 = 0 \quad (2.119)$$

We see that the surface term, if the external pressure is equal to the internal one, balances out exactly the thermal term, leaving the system with no other choice than collapsing under its own gravity. The radius will start shrinking, and very quickly, the internal pressure will rise significantly above the external one, so we can safely assume $P_{\text{ext}} = 0$ in the previous equation. The final equilibrium condition can be written as

$$\boxed{P = \frac{4\pi}{15} G\rho^2 R^2} \quad (2.120)$$

If we consider now a non-relativistic, non-degenerate Maxwell-Boltzmann gas, the pressure is given by

$$P = \rho \frac{k_B T}{m} \quad (2.121)$$

where T is the uniform temperature of the sphere. This leads to the virial condition on the gas temperature as

$$\frac{k_B T}{m} = \frac{1}{5} \frac{GM}{R} \quad (2.122)$$

where we use the definition of the total mass as

$$M = \rho \frac{4\pi}{3} R^3 \quad (2.123)$$

The problem here is that we need to know the temperature to determine the size of the star, at fixed mass. Therefore, we need to solve in parallel for the energy equation, with possibly additional heating and cooling processes within the star.

Degenerate stars

Degenerate stars are however much simpler, as the pressure depends only on density. We can therefore use the equation of state we have derived in the previous section as

$$P = A \left(\frac{\rho}{m} \right)^\Gamma = A \left(\frac{M}{m \frac{4\pi}{3} R^3} \right)^\Gamma \quad (2.124)$$

where $\Gamma = \frac{5}{3}$ in the non-relativistic case and $\Gamma = \frac{4}{3}$ in the ultra-relativistic one. We plug this into the virial relation to obtain

$$A \left(\frac{M}{m \frac{4\pi}{3} R^3} \right)^{\Gamma-2} = \frac{4\pi}{15} G m^2 R^2 \quad (2.125)$$

or, after some manipulations

$$M = \frac{4\pi}{3} m \left(\frac{4\pi G m^2}{15A} \right)^{\frac{1}{\Gamma-2}} R^{\frac{3\Gamma-4}{\Gamma-2}} \quad (2.126)$$

In the non-relativistic case, we obtain a possible equilibrium configuration if the system follows the proper mass-radius relation $M \propto R^{-3}$ or strikingly $MV = \text{constant}$. In other words, for a given mass, there is always a radius that the star can reach (by contracting or expanding) that corresponds to an equilibrium solution. In the ultra-relativistic case, however, we see that the power law vanishes and the only equilibrium solution is for a single mass. Plugging in the constant A we derived in the last chapter for the ultra-relativistic, ultra-degenerate case, we get the *Chandrasekhar mass*

$$M_{\text{eq}} = \frac{4\pi}{3m^2} \left(\frac{15hc}{16\pi G} \right)^{\frac{3}{2}} \left(\frac{3}{4\pi g} \right)^{\frac{1}{2}} \simeq 1.5 M_\odot \quad (2.127)$$

which depends only on fundamental constants. We used for m twice the proton mass $m_p = 1.66 \times 10^{-24} \text{g}$ to account for elements heavier than helium and the degeneracy parameter $g = 2$. A more accurate computation gives $M_{\text{eq}} \simeq 1.4 M_\odot$, quite close to the present derivation.

Turbulent molecular clouds

We now consider very different objects than white dwarfs or neutrons stars: molecular clouds in the interstellar medium. We simplify their description assuming they are spherical and uniform. We also use the observational fact that they are *turbulent*. We will define turbulence rigorously in the following chapters, but for the time being, we consider that the kinetic energy term $K \neq 0$ because of macroscopic random motions within the cloud. We write the kinetic term as

$$K = \int_V \frac{1}{2} \rho v^2 dV = \frac{1}{2} M (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) = \frac{3}{2} M \sigma_{1D}^2 \quad (2.128)$$

where σ_x is the velocity dispersion of the turbulence in the x-direction. The last equation assumes that the turbulence is isotropic so that each direction sees the same dispersion, noted σ_{1D} . Note that the 1D turbulence is precisely what is measured during astronomical observations, as absorption or emission lines give access to line of sight broadening due to turbulence. The observed turbulence in the interstellar medium is supersonic. This means that $\sigma_{1D} \gg a$, where a is the isothermal sound speed, related to the isothermal temperature of the ISM by

$$a = \sqrt{\frac{k_B T_0}{m}} \quad (2.129)$$

Since $T_0 \simeq 10$ K in the ISM, and the Hydrogen molecule mass is $m = 2m_p$, we have $a \simeq 0.2$ km/sec. The measured velocity dispersion in molecular cloud is much higher with $\sigma_{1D} \simeq 1 - 5$ km/sec. In the virial analysis, we can therefore safely ignore the thermal term, and define

$$\alpha_{\text{vir}} = -\frac{2K}{V} = \frac{5R\sigma_{1D}^2}{GM} \quad (2.130)$$

If $\alpha_{\text{vir}} < 1$, the cloud is collapsing, while for $\alpha_{\text{vir}} > 1$, it is expanding. Equilibrium is obtained for $\alpha_{\text{vir}} = 1$. Observed molecular clouds have $\alpha_{\text{vir}} \simeq 1 - 3$, which seems to indicate that they are slightly super-virial. The fact that we do not see any cloud with $\alpha_{\text{vir}} < 1$ is consistent with the idea that they are unstable, so they collapse quickly. We do see however many clouds with $\alpha_{\text{vir}} > 1$. They are probably slowly expanding, or we have to take into account the surface term S that may stabilize them.

2.4.2 Hydrostatic equation for spherical systems

When the system is in static equilibrium, the velocity is zero everywhere. As a consequence, gravity and pressure forces must balance each other exactly. The system is said to be in *hydrostatic equilibrium*. Such solutions are found using the Euler and the Poisson equations. We assume here again that we have a strict spherical symmetry, but this time, the system is not uniform but has an internal structure. We are seeking for a solution in one dimension, that depends only on the radial coordinate, so that

$$\rho = \rho(r), \quad P = P(r), \quad \phi = \phi(r) \quad (2.131)$$

Because $v = 0$, the mass and the energy conservation equations are automatically satisfied and all centrifugal terms are zero. The momentum conservation equation, written in Lagrangian form, becomes

$$\frac{Dv_r}{Dt} = 0 = -\frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{\partial \phi}{\partial r} \quad (2.132)$$

Using the Poisson equation in spherical coordinates, we have

$$\Delta \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = 4\pi G \rho \quad (2.133)$$

This leads to a system of two equations that describe our hydrostatic equilibrium

$$\frac{1}{\rho} \frac{\partial P}{\partial r} = -\frac{\partial \phi}{\partial r} \quad (2.134)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = 4\pi G \rho \quad (2.135)$$

To be able to solve this system of three unknown variables with only two equations, we need an equation of state for the pressure.

Lane-Emden equation for polytropes

We will study here the internal structure of stellar polytropes. We use the following general form for our polytropic EoS

$$P = P_0 \left(\frac{\rho}{\rho_0} \right)^\Gamma \quad (2.136)$$

where P_0 and ρ_0 will be defined later. Replacing the gradient of the potential in the Poisson equation by the gradient of the pressure, we obtain

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{r^2}{\rho} \frac{\partial P}{\partial r} \right) = -4\pi G \rho \quad (2.137)$$

Now plugging in the EoS, we obtain the following hydrostatic equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \Gamma \frac{P}{\rho^2} \frac{\partial \rho}{\partial r} \right) = -4\pi G \rho \quad (2.138)$$

This equation is central to the study of stellar interiors. We will now perform a change of variable to obtain a dimensionless equation. Introduction x and θ as

$$x = \frac{r}{r_0}, \quad \rho = \rho_0 \theta^n, \quad P = P_0 \theta^{n+1} \quad (2.139)$$

where the parameter n is defined as

$$\Gamma = 1 + \frac{1}{n} \quad (2.140)$$

and r_0 , the Lane-Emden radius, is given by

$$r_0^2 = \frac{n+1}{4\pi G} \frac{P_0}{\rho_0^2} \quad (2.141)$$

Using these new variables, we get the following second-order, non-linear ordinary differential equation (ODE) called the *Lane-Emden equation*

$$\boxed{\frac{1}{x^2} \frac{\partial}{\partial x} \left(x^2 \frac{\partial \theta}{\partial x} \right) = -\theta^n} \quad (2.142)$$

In the general case, for any value of n , this equation has to be solved numerically, with boundary conditions

$$\theta(0) = 1 \quad \text{and} \quad \theta(0)' = 0 \quad (2.143)$$

These solutions corresponds to a central density equal to ρ_0 . The other variable P_0 is the corresponding polytropic pressure, computed as

$$P_0 = A \left(\frac{\rho_0}{m} \right)^{\Gamma} \quad (2.144)$$

The Lane-Emden radius is then fully specified. An interesting limiting case is $n = 5$, for which the analytical solution is (left to the reader as an exercise)

$$\theta(x) = \left(1 + \frac{x^2}{3} \right)^{-1/2} \quad \text{and} \quad \rho(x) = \left(1 + \frac{x^2}{3} \right)^{-5/2} \quad (2.145)$$

This solution is typical of hydrostatic polytropes, with a central, constant density core of size $r_{\text{core}} = \sqrt{3}r_0$, surrounded by an envelope with a very steep power law density profile $\rho \propto r^{-5}$.

Bonnor-Ebert sphere for an isothermal gas

We now consider the special case of molecular clouds in the interstellar medium. We have seen that these objects are dominated by turbulence. In the densest regions of the clouds, where turbulence dissipates, local collapse is possible and molecular cores are forming. These cores are very difficult to observe because they are completely obscured by dust. Nevertheless, they appear as quasi-hydrostatic structures and are likely to be the sites of star formation. They are stellar nurseries.

The thermal state of the interstellar medium is governed by cooling and heating processes. We will see in the chapter on radiation processes how to compute these extra terms that appear in the energy equation, written here in Lagrangian form.

$$\rho \frac{D\epsilon}{Dt} = -P\nabla \cdot \mathbf{v} + n\Gamma_0 - n^2\Lambda(T) \quad (2.146)$$

Γ_0 is called the heating function and depends on the local radiation field, assumed to be constant at a given location in the Galaxy. Λ is the cooling function and depends on the temperature. The heating rate is proportional to the gas density because it features collision between photons and atoms, while the cooling rate scales as the square of the density because it is due to inelastic collisions between two atoms or two molecules. In the ISM, especially at higher density, both terms can dominate over the $p dV$ term and the gas is in *thermal equilibrium*, defined by

$$\Lambda(T) = \frac{\Gamma_0}{n} \rightarrow T = T_{\text{eq}}(\rho) \quad (2.147)$$

We see that the temperature is fixed by a thermostat, and the gas pressure (or its equation of state) becomes independent on temperature as

$$P = \rho \frac{k_B T_{\text{eq}}(\rho)}{m} \quad (2.148)$$

At densities larger than 100 H/cc, inside molecular clouds, we have $T_{\text{eq}} \simeq 10$ K and the gas becomes isothermal.

In this case, for sake of simplicity, we define the pressure by

$$P = \rho a^2 \quad \text{where} \quad a = \sqrt{\frac{k_B T_0}{m}} \quad (2.149)$$

where a is called the *isothermal sound speed*. The name will become clearer in the next sections. In this case, the hydrostatic equation becomes

$$a^2 \frac{1}{\rho} \frac{\partial \rho}{\partial r} = -\frac{\partial \phi}{\partial r} \quad (2.150)$$

and we can inject this simple form for the gradient of the potential in Poisson equation

$$\frac{a^2}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right) = -4\pi G \rho \quad (2.151)$$

We now perform a change of variables, using

$$x = \frac{r}{r_0} \quad \text{and} \quad \theta = \ln \frac{\rho}{\rho_0} \quad (2.152)$$

to obtain the Lane-Emden equation for isothermal spheres, which is

$$\boxed{\frac{1}{x^2} \frac{\partial}{\partial x} \left(x^2 \frac{\partial \theta}{\partial x} \right) = -e^\theta} \quad (2.153)$$

ρ_0 is the central density of the isothermal core, and the *Bonnor-Ebert radius* r_0 is given by

$$r_0 = \frac{a}{\sqrt{4\pi G\rho_0}} \quad (2.154)$$

We will see in the next sections that this is also the Jeans length calculated at the central core density. The previous ODE has to be solved numerically, with the non-singular boundary conditions

$$\theta(0) = 1 \quad \text{and} \quad \theta'(0) = 0 \quad (2.155)$$

The solution is called the *Bonnor-Ebert sphere* and is parametrized by the central density ρ_0 .

Singular isothermal sphere

Another solution of the same equation can be found if one allows the density profile to be singular at the origin. We use the following ansatz for the solution

$$\rho = Ar^{-\alpha} \quad \text{so that} \quad \frac{1}{\rho} \frac{\partial \rho}{\partial r} = -\frac{\alpha}{r} \quad (2.156)$$

Injecting this into the Poisson equation, we get

$$\frac{a^2}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\alpha}{r} \right) = \alpha \frac{a^2}{r^2} = 4\pi G A r^{-\alpha} \quad (2.157)$$

The only viable solution is for $\alpha = 2$ and $2\pi G A = a^2$. This gives the *singular isothermal sphere* (SIS) solution as

$$\rho(r) = \frac{a^2}{2\pi G r^2} \quad (2.158)$$

The previous Bonnor-Ebert solutions all converge to the SIS solution at large radii. The SIS is therefore a good model to describe dense molecular cores, far from their centers. It depends on only one parameter, the isothermal sound speed a , while the Bonnor-Ebert sphere also depends on the adopted central density. Note that both models, the Bonnor-Ebert sphere and the SIS, must be truncated at some maximum radius, usually where the density profile reaches the mean density of the parent molecular cloud. The model is then called a *truncated* Bonnor-Ebert or singular isothermal sphere.

2.5 Accretion disks

In this section, we study accretion disks. Because of angular momentum conservation, many astrophysical objects settle into a centrifugally supported disk after an initial phase of collapse. Popular disks in astrophysics are galactic disks, circumstellar protoplanetary disks and disks around black holes. Their physical properties are very different but they share the common notion of being in a centrifugal equilibrium. Disks are usually considered to be axisymmetric. They are invariant under a rotation around their rotation axis. The natural coordinate system is therefore the cylindrical coordinate system. Moreover, in this section, we consider disks that are not self-gravitating. Self-gravitating disks will be one of the main topics of the last chapter of this course. Galactic disks, for example, are self-gravitating. Circumstellar disks or disks around black holes, on the other hand, are dominated by the gravity of their central object, considered as a point mass at the origin of the coordinate system, so that we have

$$\mathbf{g} = g_r \mathbf{e}_r + g_z \mathbf{e}_z \quad \text{with} \quad g_z = -\frac{GM_*}{R^3} z \quad \text{and} \quad g_r = -\frac{GM_*}{R^3} r \quad (2.159)$$

We have introduced the spherical radius R , related to our cylindrical space coordinates as $R^2 = r^2 + z^2$. Because of axisymmetry, we consider that the disk density profile depends only on r and z . Moreover, we assume that the vertical velocity is zero, so that we have vertical hydrostatic equilibrium $v_z = 0$. The radial and tangential velocities are however non-zero, as we allow for rotation and inflow of matter towards the central object. We finally assume that $v_r(r)$ and $v_\theta(r)$ depends only on the radial coordinate, not on z . This is called a *laminar disk*.

Using the previously derived Euler equations in cylindrical coordinates, we can write the mass conservation equation as

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) = 0 \quad (2.160)$$

where we used the fact that all partial derivatives with respect to θ are zero and that $v_z = 0$. We introduce the *surface density*, a fundamental quantity when describing disks and defined as

$$\Sigma(r) = \int_{-\infty}^{+\infty} \rho(r, z) dz \quad (2.161)$$

Integrating the mass conservation equation over z ,

$$\int \frac{\partial \rho}{\partial t} dz + \int \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) dz = 0 \quad (2.162)$$

and using the fact that t , r and z are independent, we obtain a new form of the mass conservation equation

$$\boxed{\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) = 0} \quad (2.163)$$

The total disk mass can be obtained by integrating the surface density over the entire disk as

$$M_{\text{disk}} = \int_0^{+\infty} \int_0^{2\pi} \Sigma(r) r dr d\theta = \int_0^{+\infty} \Sigma(r) 2\pi r dr \quad (2.164)$$

where we have exploited the axisymmetry of the disk and integrated out the polar angle θ . We can also define the mass accretion rate as the net flux through a circle of radius r towards the star as

$$\dot{M}(r) = \Sigma 2\pi r v_r \quad (2.165)$$

We see that in order to have accretion of matter onto the star, the radial velocity cannot be zero, it has to be negative. We also see from the mass conservation equation that a stationary solution with $\frac{\partial \Sigma}{\partial t} = 0$ implies $\dot{M} = \text{constant}$, and the constant has to be also negative.

The momentum conservation equation in cylindrical coordinates writes in the radial direction as

$$\rho \left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} \right] = -\frac{\partial P}{\partial r} - \rho \frac{GM_*}{R^3} r \quad (2.166)$$

We again integrate the equation in the z -direction and use the fact that v_r and v_θ are independent of z . We also assume the disk is thin, so we have $R^2 = r^2 + z^2 \simeq r^2$.

$$\Sigma \left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} \right] = -\frac{\partial \Pi}{\partial r} - \Sigma \frac{GM_*}{r^2} \quad (2.167)$$

The equation of state of the gas is considered as *locally isothermal*, so that

$$P = \rho a^2(r) \quad \text{and} \quad \Pi = \Sigma a^2(r) \quad (2.168)$$

The gas temperature is determined by the thermal balance between various heating and cooling processes in the disk, and will be determined elsewhere. The only assumption is that the gas temperature depends on the radial coordinate only. The tangential momentum conservation equation writes similarly as

$$\rho \left[\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} \right] = 0 \quad (2.169)$$

Indeed, we have no pressure gradients in the θ -direction and the gravity is purely radial.

$$\Sigma \left[\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} \right] = 0 \quad (2.170)$$

We conclude that we have coupled three differential equations with three unknown variables Σ , v_r and v_θ that we can solve, knowing some initial conditions and proper boundary conditions.

2.5.1 Centrifugal equilibrium solution

We are looking for stationary solutions with all $\frac{\partial}{\partial t} = 0$. We are also imposing $v_r = 0$ which correspond to a rotating disk with zero accretion rate. The mass conservation equation and the tangential momentum equation are both trivially satisfied. We get however in the radial direction the following constraint

$$\frac{v_\theta^2}{r} - \frac{GM_*}{r^2} = \frac{1}{\Sigma} \frac{\partial \Pi}{\partial r} \quad (2.171)$$

The first term on the left-hand side of the equations is the centrifugal force. The second term is the gravity of the central object, while the right-hand side is the pressure gradient. We define a *cold disk* if $a(r) \ll v_\theta$ so we can neglect the pressure gradient. In this case, centrifugal forces balance exactly gravity and we obtain a *Keplerian disk*. In this case, the tangential speed is given by

$$v_\theta \simeq \sqrt{\frac{GM_*}{r}} \equiv v_K \quad (2.172)$$

where the latter is called the Keplerian velocity. It is convenient to define the orbital frequency as

$$\Omega = \frac{v_\theta}{r} \quad (2.173)$$

so that the orbital period writes

$$T_{\text{orb}} = \frac{2\pi}{\Omega} \quad (2.174)$$

2.5.2 Viscous disks

Observed accretion disks, around protostars or around black holes, appear very luminous. It means that there must be a energy dissipation mechanism for which heat is created and radiated away towards an observer. What we have learned so far is that centrifugally supported disks feature *differential rotation*. It means that adjacent rings of gas rotates at different velocities, imposed by the Keplerian motions, which lead to strong shear. We know from the last chapter that strong shear creates strong viscous stresses. In our case, this lead to a viscous force trying to restore a situation without shear. We expect therefore a tangential force to be present. As a consequence, by slowing down adjacent rings, we will create radial motions and a non-zero accretion rate. It is this model of viscous disks that we now want to explore.

$$\text{We assume } v_r \neq 0 \quad \text{but} \quad v_r \ll v_\theta \quad (2.175)$$

Recall that the shear tensor is defined by

$$\mathbb{S} = \mathbb{G} + \mathbb{G}^T - \frac{2}{3} (\nabla \cdot \mathbf{v}) \mathbb{I} \quad (2.176)$$

Neglecting all terms depending on v_r , we find

$$\mathbb{S} = \begin{bmatrix} 0 & r \frac{\partial \Omega}{\partial r} & 0 \\ r \frac{\partial \Omega}{\partial r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.177)$$

We see that solid body rotation, defined as $\Omega = \text{constant}$, does not trigger viscous stresses. On the other hand, differential rotation, for which Ω is strongly varying with radius, does lead to strong shear flows. The viscous force has only a component in the tangential direction that writes

$$(\nabla \cdot \mathbb{S})_\theta = \frac{\partial S_{r\theta}}{\partial r} + 2 \frac{S_{r\theta}}{r} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 S_{r\theta}) \quad (2.178)$$

Multiplying \mathbb{S} by the viscosity coefficient μ , we obtain the momentum conservation equation in the tangential direction

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\mu r^3 \frac{\partial \Omega}{\partial r} \right) \quad (2.179)$$

We introduce the diffusion coefficient $\nu(r) = \mu/\rho$ and assume for simplicity that it is only a function of r . We then integrate the previous equation along the z -axis and introduce the specific angular momentum $l = r v_\theta$. We finally obtain the angular momentum conservation equation

$$\boxed{\Sigma \left(\frac{\partial}{\partial t} (r v_\theta) + v_r \frac{\partial}{\partial r} (r v_\theta) \right) = \frac{1}{r} \frac{\partial}{\partial r} \left(\nu \Sigma r^3 \frac{\partial \Omega}{\partial r} \right)} \quad (2.180)$$

where we used the following trick that allowed us to absorb the centrifugal term

$$v_r \left[\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \right] = \frac{v_r}{r} \frac{\partial}{\partial r} (r v_\theta) \quad (2.181)$$

This equation nicely expresses the fact that viscosity transports angular momentum. Without viscosity, angular momentum is strictly conserved. Viscosity acts as a (non-trivial) diffusion operator, which will transport angular momentum outside, while mass will be transported inside, towards the central star.

2.5.3 Stationary viscous disk solution

As usual, we look for solutions for which all $\frac{\partial}{\partial t} = 0$, but this time we allow for $v_r \neq 0$ owing to the viscous stresses. The angular momentum conservation equation can be written as

$$\Sigma v_r \frac{\partial}{\partial r}(r v_\theta) = \frac{1}{r} \frac{\partial}{\partial r} \left(\nu \Sigma r^3 \frac{\partial \Omega}{\partial r} \right) \quad (2.182)$$

Now, consider our quasi-Keplerian disk model for which

$$v_\theta \simeq \sqrt{\frac{GM_*}{r}} \propto r^{-1/2} \quad \text{and} \quad \Omega \simeq \sqrt{\frac{GM_*}{r^3}} \propto r^{-3/2} \quad (2.183)$$

Injecting these dependencies in the angular momentum conservation equation, we find the radial velocity as

$$v_r = -\frac{3}{\Sigma r^{1/2}} \frac{\partial}{\partial r} (\nu \Sigma r^{1/2}) \quad (2.184)$$

and the corresponding accretion rate as

$$\dot{M} = -6\pi r^{1/2} \frac{\partial}{\partial r} (\nu \Sigma r^{1/2}) \quad (2.185)$$

As already discussed, if we want the solution to be stationary, then the accretion rate has to be constant. We can easily integrate the previous differential equation and find a solution for the disk surface density as

$$\Sigma = -\frac{\dot{M}}{3\pi\nu} \left(1 - \sqrt{\frac{r_0}{r}} \right) \quad (2.186)$$

where r_0 is the radius where $\Sigma = 0$ and the solution is valid only for $r > r_0$. Note that r_0 is also the radius at which $v_r \rightarrow -\infty$. The only physical solution would be to put this singularity at the exact location of the star, namely $r_0 = 0$. This leads to the final solution

$$\Sigma(r) = -\frac{\dot{M}}{3\pi\nu(r)} \quad (2.187)$$

The solution for the gas surface density depends therefore directly on the inverse of the diffusion coefficient. Assuming that ν is a constant, we can integrate this constant surface density solution and connect the accretion rate to the total disk mass as

$$\dot{M} = -3\nu \frac{M_{\text{disk}}}{r_{\text{disk}}^2} \quad (2.188)$$

Let's now consider typical values for observed protoplanetary disks

$$M_{\text{disk}} = 10^{-2} M_\odot \quad \text{and} \quad r_{\text{disk}} = 10 \text{ AU} \quad (2.189)$$

Let's assume that we have a relatively thin disk (see the next section) with a thickness of 1 AU. The mean gas density in the disk will thus be around $n \simeq 10^{13}$ H/cc. Using the typical hard-sphere cross section of 10^{-15} cm², we get for the mean free path $\lambda_{\text{coll}} \simeq 1$ m. With a gas temperature of 300 K corresponding to a sound speed of 1 km/s, we get

$$\nu \simeq 10^7 \text{ cm}^2 \text{ s}^{-1} \quad \text{and} \quad \dot{M} \simeq -10^{-15} M_\odot / \text{yr} \quad (2.190)$$

It is possible to compute the luminosity of the accretion disk, assuming balance between viscous dissipation and radiative cooling (left to the reader as an exercise). We can infer from the

observed luminosity (in the infrared band) the accretion rate around protostars. We usually observe \dot{M} between $10^{-7}M_{\odot}/\text{yr}$ and $10^{-6}M_{\odot}/\text{yr}$, eight to nine orders of magnitude larger than our naive estimate based on molecular viscosity. The same is true for accretion disks around black holes, with very large observed luminosity in the ultraviolet, far larger than what is predicted with ions viscosity.

How do we reconcile the theory of viscous disks with observations? One possible explanation is turbulence. Accretion disks are probably not laminar, as assumed here, but turbulent. We will describe the theory of turbulent flows in the next sections. The main conclusion will be that the effects of turbulence can be modelled as an effective viscous stress, called the turbulent stress (or the Reynolds stress), with an effective diffusion coefficient

$$\nu_{\text{T}} = \ell_{\text{T}}\sigma_{\text{T}} \quad (2.191)$$

where ℓ_{T} is the size of the largest eddies in the disk at any given radius, and σ_{T} is the velocity dispersion of the turbulent eddies. Depending on the exact nature of the turbulence, the size of the eddies is considered to be a fraction of the disk scale height and their velocity dispersion a fraction of the sound speed. These fractions are encoded in the so-called α -disk model with a turbulent viscosity diffusion coefficient written as

$$\nu_{\text{T}} = \alpha H(r)a(r) \quad \text{where} \quad \alpha \simeq 1\% \quad (2.192)$$

Using $H \simeq 1 \text{ AU}$ and $a \simeq 1 \text{ km/s}$, we get

$$\nu_{\text{T}} \simeq 10^{16} \text{ cm}^2\text{s}^{-1} \quad \text{and} \quad \dot{M} \simeq -10^{-6} M_{\odot}/\text{yr} \quad (2.193)$$

in much better agreement with observations. We now know that accretion disks around black holes are indeed turbulent. The turbulence is maintained by the magnetic field through a process called magneto-rotational instability. Since the ALMA observatory has been in operation, we also know that protostellar disks are not turbulent. So the α -disk model seems to be wrong in this case. The origin of the accretion in proto-planetary disk is therefore still an unsolved mystery.

2.5.4 Vertical equilibrium

One last aspect of accretion disks that we have not discussed yet is their vertical structure. In the previous sections, we have assumed that the vertical velocity is zero. This is only possible if the fluid satisfies the vertical hydrostatic equilibrium written as

$$\frac{\partial P}{\partial z} = -\rho \frac{GM}{R^3} z \quad (2.194)$$

with $R^2 = r^2 + z^2$. In the thin Keplerian disk approximation $z \ll r$ and we get

$$\frac{1}{\rho} \frac{\partial \rho}{\partial z} = -\frac{GM}{r^3 a^2} z = -\frac{\Omega^2}{a^2} z \quad (2.195)$$

Using our equation of state $P = \rho a^2(r)$, we can integrate the equation and obtain the vertical density profile has

$$\rho = \rho_0 \exp\left(-\frac{\Omega^2 z^2}{a^2}\right) = \rho_0 \exp\left(-\frac{z^2}{2H^2}\right) \quad (2.196)$$

where ρ_0 is the midplane density and the disk scale height H is given by

$$H(r) = \frac{a(r)}{\Omega(r)} \quad (2.197)$$

We see that if the disk is cold, then it is also thin. Integrating the Gaussian profile vertically, we find $\Sigma = \sqrt{2\pi}\rho_0 H$ so that the vertical density profile is fully determined by

$$\rho(r, z) = \frac{\Sigma(r)}{\sqrt{2\pi H(r)^2}} \exp\left(-\frac{z^2}{2H(r)^2}\right) \quad (2.198)$$

Since we now know the disk scale height, we can also compute the turbulent diffusion coefficient as

$$\nu_T = \alpha \frac{a(r)^2}{\Omega(r)} \quad (2.199)$$

Interestingly enough, for a strictly isothermal disk $a(r) = a_0$ constant, this leads to the following disk surface density profile for the viscous, stationary and quasi-Keplerian solution

$$\Sigma(r) = -\frac{\dot{M}\sqrt{GM_*}}{3\pi\alpha a_0^2} r^{-3/2} \quad (2.200)$$

Note the power law $-3/2$ which compares almost perfectly with the Minimum Mass Solar Nebula, the disk model one would infer by looking at the mass distribution of planets in our present day solar system.

2.6 Spherical Bondi accretion flow

We now discuss a very famous stationary solution in astrophysics, namely the Bondi accretion model. It describes how gas can be accreted from a uniform background to a point mass, either a star or a black hole. The background fluid is at rest, so that the problem is perfectly spherically symmetric. This is however a non-static configuration, as the fluid is allowed to flow towards the central object. Stationary solutions are quite common in fluid dynamics. Famous examples are Poiseuille flow, which describes a laminar viscous fluid flowing through a pipe, Couette flow, flow past a cylinder, isentropic nozzle flow, etc. These flows are all stationary and their main properties are driven by a famous theorem, called *Bernoulli theorem*. We will first derive the two different forms of this theorem, before we embark in the description of the Bondi flow.

2.6.1 First Bernoulli theorem

As always, in order to simplify the theoretical description of the fluid, we are looking for a conserved quantity that we can exploit in our calculation. For stationary flows, this quantity is the *enthalpy*. We define the specific enthalpy using the usual thermodynamics definition

$$h = \epsilon + \frac{P}{\rho} \quad (2.201)$$

We will now show that this quantity, together with the kinetic and potential energy, is conserved in a certain sense in a stationary flow. We first re-write the Euler equations in Lagrangian form as

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P - \rho \nabla \phi \quad (2.202)$$

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{v} \quad (2.203)$$

$$\frac{D\epsilon}{Dt} = -\frac{P}{\rho} \nabla \cdot \mathbf{v} \quad (2.204)$$

We want to compute the time evolution of the *total specific enthalpy* defined as

$$H = h + \frac{v^2}{2} + \phi \quad (2.205)$$

We first take the Lagrange derivative of the specific kinetic energy

$$\frac{D}{Dt} \left(\frac{v^2}{2} \right) = \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = -\frac{\mathbf{v}}{\rho} \cdot \nabla P - \mathbf{v} \cdot \nabla \phi \quad (2.206)$$

We then compute the Lagrange derivative of the potential

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi \quad (2.207)$$

and finally, we compute the Lagrange derivative of the enthalpy, which is slightly more involved as

$$\frac{Dh}{Dt} = -\frac{P}{\rho} \nabla \mathbf{v} + \frac{1}{\rho} \frac{DP}{Dt} - \frac{P}{\rho^2} \frac{D\rho}{Dt} \quad (2.208)$$

We now use the mass conservation equation and the Lagrange derivative of the pressure

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P \quad (2.209)$$

Summing up all these equations leads to the first Bernoulli theorem

$$\boxed{\frac{D}{Dt} \left(\frac{v^2}{2} + \phi + h \right) = \frac{\partial \phi}{\partial t} + \frac{1}{\rho} \frac{\partial P}{\partial t}} \quad (2.210)$$

We immediately see that for a stationary flow, the right-hand side vanishes and the Lagrange derivative is zero. For a stationary flow, the trajectories of the fluid elements are called *streamlines*. The first Bernoulli theorem tells us that the total specific enthalpy is constant along streamlines. A very important precision here: each streamline will have a different constant. This constant is called *the Bernoulli constant*. This theorem is very general: no assumption on the flow properties (except stationarity) or on the equation of state have been made. But it is not so powerful, in the sense that we have now a family of streamlines, each one of them with a different Bernoulli constant, determined by the boundary conditions. We now discuss the second Bernoulli theorem, a much more powerful result, with however an important restriction.

2.6.2 Second Bernoulli theorem

The second Bernoulli theorem is entirely built on a rather complex vector calculus identity. We start from the Euler equation in Lagrangian form

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \phi \quad (2.211)$$

and we use the following vector identity

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \left(\frac{v^2}{2} \right) - \mathbf{v} \times \boldsymbol{\omega} \quad (2.212)$$

Here we introduced the *vorticity* of the flow, namely the curl of the velocity field

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \quad (2.213)$$

We also introduce another new quantity Π defined as

$$d\Pi = \frac{1}{\rho} dP \quad (2.214)$$

This quantity is defined for incompressible flows, for which ρ is constant and thus $\Pi = P/\rho$ or for polytropes where $P = P(\rho)$ and we can integrate the previous form. For general equation of states, however, this quantity can be tricky to compute, and additional conditions have to be set, such as the uniformity of the entropy for example. We assume here that the fluid properties are such that this quantity can be computed. We finally obtain the second Bernoulli theorem as

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{v^2}{2} + \phi + \Pi \right) = \mathbf{v} \times \boldsymbol{\omega}} \quad (2.215)$$

We will see in the next sections that we can derive a new equation called the *vorticity equation* which, for the incompressible or polytropic fluids we consider here can be written as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = \nu \Delta \boldsymbol{\omega} \quad (2.216)$$

We see that, if we wait long enough for the flow to become stationary and if there are no source of vorticity in the system, the fluid vorticity will decay due to viscous processes and ultimately vanishes. We deduce from the second Bernoulli theorem that, for a stationary and curl-free flow for which $\boldsymbol{\omega} = 0$ everywhere, we have

$$\frac{v^2}{2} + \phi + \Pi \quad \text{uniform everywhere.} \quad (2.217)$$

This is a very powerful result, but it comes with a prize: We can only apply it to polytropic fluids and curl-free conditions, which are not always satisfied in the Universe.

2.6.3 Bondi solution

We now move to the specific problem at hand, namely a spherical stationary flow towards a point mass. Before deriving the exact solution, we would like to explore a very simple model that can be used to quickly derive the main quantity that defines the accretion flow: the accretion rate. For this, we consider our star or black hole as a static point mass sitting at the origin of our spherical coordinate system. We now consider particles moving towards the star, with velocity v and impact parameter b . Using a simple binding energy argument, we can separate between bound particles that will be accreted by the star and unbound particles that will be deflected but will survive the encounter and leave the scene by computing the zero binding energy condition as

$$\frac{1}{2}mv^2 - m\frac{GM_*}{b} = 0 \quad (2.218)$$

This defines the Bondi radius as the dividing line between bound and unbound particles

$$r_B = \frac{2GM_*}{v^2} \quad (2.219)$$

We now compute the accretion rate of particles with velocity v using the collision cylinder that we used in kinetic theory and pick for the cross section a disk of radius equal to the Bondi radius

$$\dot{M}_{\text{acc}} = \rho \pi r_B^2 v = 4\pi \rho \frac{(GM_*)^2}{v^3} \quad (2.220)$$

This accretion rate is only for particles of velocity v . We now need to compute the accretion rate for all possible particle velocities following a Maxwell-Boltzmann distribution. For this, we brute force replace v by a , the microscopic velocity dispersion $a = \sqrt{\frac{k_B T}{m}}$, and obtain

$$\boxed{\dot{M}_{\text{acc}} = 4\pi\rho\frac{G^2 M_*^2}{a^3}} \quad (2.221)$$

This very rough derivation is very approximate, but it contains most of the physics of the problem. This is the one I encourage you to remember. As we will see now, it is also surprisingly close to the exact solution.

Let's now define more precisely the problem we would like to solve. We assume that the fluid is at rest at infinity, with a uniform density and pressure. We also assume that the gas is isothermal, so that $P = \rho a^2$.

$$\text{At } r \rightarrow +\infty, \text{ we have } \rho \rightarrow \rho_\infty, \quad P \rightarrow P_\infty = \rho_\infty a^2, \quad \text{and } v \rightarrow 0. \quad (2.222)$$

We are looking for a spherically symmetric solution with the two following unknown variables $\rho(r)$ and $v(r)$. For the latter, we dropped the index r as this is the only non-zero velocity component. The mass conservation equation in spherical coordinates writes in this case

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0 \quad (2.223)$$

For a stationary solution, we thus have a constant accretion rate, defined as the total flux through a sphere of radius r as

$$\dot{M} = 4\pi r^2 \rho v \quad (2.224)$$

Since our fluid is isothermal (a special case of a polytrope) and since we consider that after some time the vorticity vanishes, owing to viscous dissipation, we can apply the second Bernoulli theorem and write

$$\frac{v^2}{2} - \frac{GM_*}{r} + \Pi = \text{constant} \quad (2.225)$$

We compute Π using the definition

$$\Pi = \int_{+\infty}^r \frac{dP}{\rho} = a^2 \int_{+\infty}^r \frac{d\rho}{\rho} = a^2 \ln \frac{\rho(r)}{\rho_\infty} \quad (2.226)$$

We can now evaluate the Bernoulli constant at $r \rightarrow +\infty$ and find the final form of Bernoulli theorem

$$\frac{v^2}{2} - \frac{GM_*}{r} + a^2 \ln \frac{\rho}{\rho_\infty} = 0 \quad (2.227)$$

As usual, we define the following dimensionless variables

$$\tilde{r} = \frac{r}{r_B}, \quad \tilde{v} = \frac{v(r)}{a}, \quad \tilde{\rho} = \frac{\rho(r)}{\rho_\infty} \quad \text{and} \quad \lambda = \frac{\dot{M}(r)}{\dot{M}_B} \quad (2.228)$$

If we choose for the Bondi radius and the Bondi accretion rate the following definitions

$$r_B = \frac{GM_*}{a^2} \quad \text{and} \quad \dot{M}_B = 4\pi r_B^2 \rho_\infty a \quad (2.229)$$

then our two equations become dimensionless, with the mass flux being

$$\lambda = \tilde{r}^2 \tilde{\rho} \tilde{v} \quad (2.230)$$

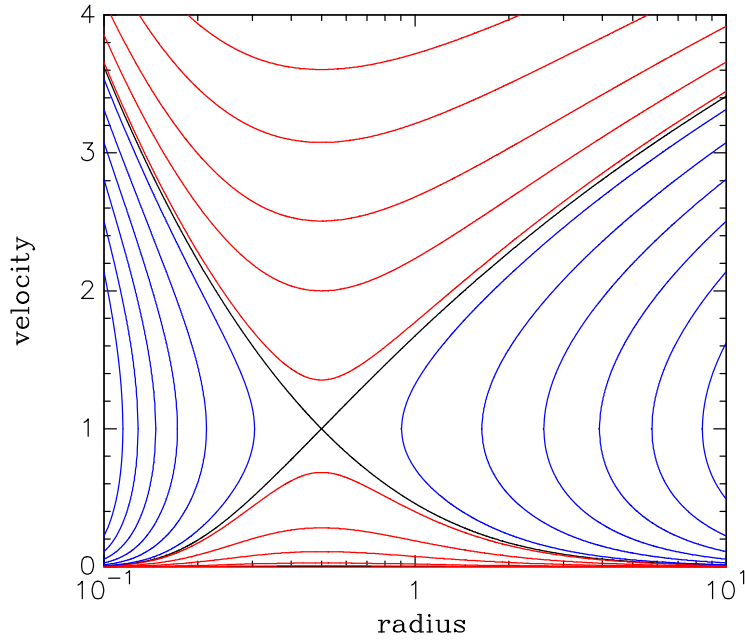


Figure 2.1: Velocity profiles for the isothermal Bondi solution with various λ . The black line corresponds to the critical value $\lambda = \lambda_{\text{crit}} \simeq 1.12$. Red lines corresponds to $\lambda < \lambda_{\text{crit}}$ and blue lines for $\lambda > \lambda_{\text{crit}}$.

and Bernoulli equation being

$$\frac{\tilde{v}^2}{2} - \frac{1}{\tilde{r}} + \ln \tilde{\rho} = 0 \quad (2.231)$$

Using the first equation, we can express $\tilde{\rho}$ as a function of the velocity and λ , and inject the expression in the second equation. We obtain a family of implicit solutions for $v(r)$ parametrised by λ (see Figure 2.1). Only the black curves corresponds to a physical solution. We have two such solutions: the one with $v \rightarrow 0$ as $r \rightarrow +\infty$ corresponds to the Bondi accretion solution, while the other one, with $v \rightarrow 0$ as $r \rightarrow 0$ corresponds to the Parker wind solution, of interest to describe stellar winds. These two solutions correspond to a critical accretion rate λ , for which the flow becomes supersonic at a critical radius, called the sonic radius.

To find the sonic radius, we differentiate the mass flux equation to get

$$2 \frac{d\tilde{r}}{\tilde{r}} + \frac{d\tilde{\rho}}{\tilde{\rho}} + \frac{d\tilde{v}}{\tilde{v}} = 0 \quad (2.232)$$

and the Bernoulli equation to get

$$\tilde{v} d\tilde{v} + \frac{d\tilde{r}}{\tilde{r}^2} + \frac{d\tilde{\rho}}{\tilde{\rho}} = 0 \quad (2.233)$$

Combining the two equations, we can subtract out the density and obtain the relation

$$\left(\tilde{v} - \frac{1}{\tilde{v}} \right) d\tilde{v} = \left(\frac{2}{\tilde{r}} - \frac{1}{\tilde{r}^2} \right) d\tilde{r} \quad (2.234)$$

which gives us the spatial derivative of the velocity field as

$$\frac{\partial \tilde{v}}{\partial \tilde{r}} = \frac{2/\tilde{r} - 1/\tilde{r}^2}{\tilde{v} - 1/\tilde{v}} \quad (2.235)$$

We see that the velocity derivative will always be singular at $\tilde{v} = 1$, which corresponds to the velocity being equal to the speed of sound, unless this happens precisely at the critical radius

$$\tilde{r}_{\text{crit}} = \frac{1}{2} \quad (2.236)$$

If we inject this value in the Bernoulli equation, we find

$$\tilde{\rho} = \exp\left(\frac{3}{2}\right) \quad (2.237)$$

and finally, using the mass flux equation, we get

$$\lambda_{\text{crit}} = \frac{1}{4} \exp\left(\frac{3}{2}\right) \simeq 1.12 \quad (2.238)$$

Finally, the exact solution for the Bondi accretion rate is

$$\dot{M} = \lambda_{\text{crit}} \dot{M}_{\text{B}} \quad (2.239)$$

quite close to our initial guess. We can examine the corresponding density profile. For $r \gg r_{\text{crit}} = r_{\text{B}}/2$, the density is almost constant and equal to ρ_{∞} . Inside the sonic radius, for $r \ll r_{\text{B}}/2$, the flow is supersonic. We can neglect the $\ln \rho$ term in the Bernoulli equation, and we obtain the following singular velocity and density profiles

$$v \simeq 2a \left(\frac{r_{\text{B}}/2}{r}\right)^{1/2} \quad \text{and} \quad \rho \simeq 2\rho_{\infty} \left(\frac{r_{\text{B}}/2}{r}\right)^{3/2} \quad (2.240)$$

This derivation is valid for an isothermal equation of state. As an exercise, we leave it to the reader to derive the Bondi solution for a polytropic equation of state, for which

$$P = P_{\infty} \left(\frac{\rho}{\rho_{\infty}}\right)^{\Gamma} \quad (2.241)$$

The solution is quite similar, with however some small interesting quantitative differences.

2.7 Sound waves

So far, we have studied static equilibrium and stationary solutions of the Euler equations with application to astrophysical fluid flows. We now need to check that these solutions are stable. The traditional approach to study the stability of equilibrium flows is to perturb slightly the solution with deviations that are small enough to allow for a *linearisation* of the Euler equations. The resulting system of equations is much simpler to solve, and lead to particular propagating solutions called *sound waves*. If the amplitude of these waves remains constant (or decays with time), the equilibrium solution is said to be stable. If, on the other hand, the amplitude increases exponentially fast with time, we have an instability and the equilibrium solution is not viable.

2.7.1 Stable propagating waves

We consider the following simple equilibrium solution of the Euler equations $\rho_{\text{eq}} = \rho_0$ constant and $v_{\text{eq}} = 0$, with an isothermal equation of state $P = \rho a^2$. We write the Euler equations in one space dimension with $\mathbf{v} = (v_x, 0, 0)$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \quad (2.242)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} = 0 \quad (2.243)$$

where we dropped the index x in the velocity variable for simplicity. There is no energy equation to consider in the isothermal case. We define the perturbed solution as

$$\rho = \rho_0 + \delta\rho(x, t) \quad (2.244)$$

$$v = 0 + \delta v(x, t) \quad (2.245)$$

with $|\delta\rho| \ll \rho_0$ and $|\delta v| \ll a$. We can now linearise the two Euler equations by substituting these expressions and neglect the quadratic terms. We also assume that our perturbations are smooth enough, so that their derivatives remain bound. For the mass conservation equation, we get

$$\frac{\partial}{\partial t}(\delta\rho) + (\rho_0 + \delta\rho)\frac{\partial}{\partial x}(\delta v) + \delta v\frac{\partial}{\partial x}(\delta\rho) = 0 \quad (2.246)$$

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_0\frac{\partial}{\partial x}(\delta v) + \delta\rho\frac{\partial}{\partial x}(\delta v) + \delta v\frac{\partial}{\partial x}(\delta\rho) = 0 \quad (2.247)$$

Neglecting the two rightmost quadratic terms in this equation, we obtain to leading order the linear equation

$$\boxed{\frac{\partial}{\partial t}(\delta\rho) + \rho_0\frac{\partial}{\partial x}(\delta v) = 0} \quad (2.248)$$

The momentum conservation equation becomes

$$\frac{\partial}{\partial t}(\delta v) + \delta v\frac{\partial}{\partial x}(\delta v) + \frac{a^2}{\rho}\frac{\partial}{\partial x}(\rho_0 + \delta\rho) = 0 \quad (2.249)$$

The last term can be Taylor-expanded as

$$\frac{1}{\rho} \simeq \frac{1}{\rho_0} \left(1 - \frac{\delta\rho}{\rho_0}\right) \quad \text{so that} \quad \frac{a^2}{\rho}\frac{\partial}{\partial x}(\delta\rho) \simeq \frac{a^2}{\rho_0}\frac{\partial}{\partial x}(\delta\rho) - \frac{a^2}{\rho_0^2}\delta\rho\frac{\partial}{\partial x}(\delta\rho) \quad (2.250)$$

Neglecting again the quadratic terms, we obtain to leading order the linear form of the momentum conservation equation

$$\boxed{\frac{\partial}{\partial t}(\delta v) + \frac{a^2}{\rho_0}\frac{\partial}{\partial x}(\delta\rho) = 0} \quad (2.251)$$

Now, if we take the time derivative of the first equation, and the space derivative of the second equation, we combine the two equations and obtain the famous *wave equation*.

$$\boxed{\frac{\partial^2}{\partial t^2}(\delta\rho) = a^2\frac{\partial^2}{\partial x^2}(\delta\rho)} \quad (2.252)$$

We are looking for planar wave solutions with the following *Ansatz*

$$\delta\rho(x, t) \equiv \Delta\rho e^{i(kx - \omega t)} \quad (2.253)$$

$$\delta v(x, t) \equiv \Delta v e^{i(kx - \omega t)} \quad (2.254)$$

where $\Delta\rho$ and Δv are the constant amplitude of the waves. Plugging this into the wave equation, we get

$$-\omega^2\Delta\rho + a^2k^2\Delta\rho = 0 \quad (2.255)$$

Since we are looking for non-trivial solutions with $\Delta\rho \neq 0$, we get *the dispersion relation* of the waves.

$$\boxed{\omega^2 = a^2k^2} \quad (2.256)$$

The wave speed is given by $\frac{\omega}{k} = \pm a$. In this case, it is called the *sound speed*. From the second linearized Euler equation, we get a relation between the amplitudes

$$-i\omega\Delta v + ik\frac{a^2}{\rho}\Delta\rho = 0 \quad (2.257)$$

so that

$$\frac{\Delta v}{a} = \pm \frac{\Delta\rho}{\rho_0} \quad (2.258)$$

The particles of the fluid are moving with the velocity field δv , they are oscillating back and forth around a fixed average position. So particles are not really moving away from the equilibrium solution. Travelling waves are moving at the speed of sound, but only as a virtual feature. They don't carry mass, only momentum and energy. Their amplitude is constant, so that the corresponding equilibrium solution is said to be stable.

2.7.2 Thermal instability

We now consider the more general case of a polytropic equation of state, for which

$$P = P(\rho) = \rho \frac{k_B T_{eq}(\rho)}{m} \quad (2.259)$$

where the temperature is determined by a strict balance between heating and cooling processes. Figure 2.2 shows the equilibrium pressure within the interstellar medium of the Milky Way at different distances from the centre of the Galaxy. Closer to the centre, the radiation field and the cosmic ray flux are both stronger, and the corresponding equilibrium pressure higher. At low density, the gas settles into an isothermal equilibrium with $T \simeq 10^4$ K. At intermediate densities, however, the pressure is dropping because cooling scales as n^2 and it takes over these various heating processes, lowering sharply the equilibrium temperature. At higher density, we reach an asymptotic isothermal regime with $T \simeq 10$ K and the pressure is rising again.

We consider here again a uniform equilibrium state with $\rho = \rho_0$, $v = 0$ and $T = T_{eq}(\rho_0)$. Starting again from the 1D Euler equations, with however a different equation of state, we get the following linearized form

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_0 \frac{\partial}{\partial x}(\delta v) = 0 \quad (2.260)$$

$$\frac{\partial}{\partial t}(\delta v) + \frac{P'(\rho_0)}{\rho_0} \frac{\partial}{\partial x}(\delta\rho) = 0 \quad (2.261)$$

with $P'(\rho) = \frac{\partial P}{\partial \rho}$. Taking the time derivative of the first equation, swapping the time and space derivative of the second term and replacing it by the the second equation, we get a new form of the wave equation

$$\frac{\partial^2}{\partial t^2}(\delta\rho) - P'(\rho_0) \frac{\partial^2}{\partial x^2}(\delta\rho) = 0 \quad (2.262)$$

We are considering here again solutions of the form $\delta\rho = \Delta\rho e^{i(kx - \omega t)}$, where k is real but ω can now be a complex number. Injecting the solution in the second order linear differential equation leads to the the dispersion relation

$$\omega^2 = P'(\rho_0)k^2 \quad (2.263)$$

Note that we can recover the isothermal case easily because for $P = \rho a^2$, $P' = a^2$ and the dispersion relation leads to a real wave frequency with wave velocity $\frac{\omega}{k} = \pm a$. In the general polytropic case, however, we have to distinguish two different cases:

- $P'(\rho_0) > 0$

In this case, both k and ω are real and we can define the *sound speed* as $c_s^2 = P'(\rho_0)$ or

$$c_s = \sqrt{P'(\rho_0)} \quad (2.264)$$

We have propagating waves with a constant amplitude and wave velocities $\pm c_s$. Note that if we include viscosity and/or thermal conduction, the wave amplitude will eventually slowly decay. In conclusion, it means our equilibrium solution is *stable*.

- $P'(\rho_0) < 0$

In this case, we can find proper solutions only if k is real and ω is purely imaginary, because $\omega^2 < 0$. We define the growth rate $\gamma \in \mathbb{R}$ as $\omega = i\gamma$. The solutions are then defined by

$$\gamma = \pm \sqrt{-P'(\rho_0)}k \quad (2.265)$$

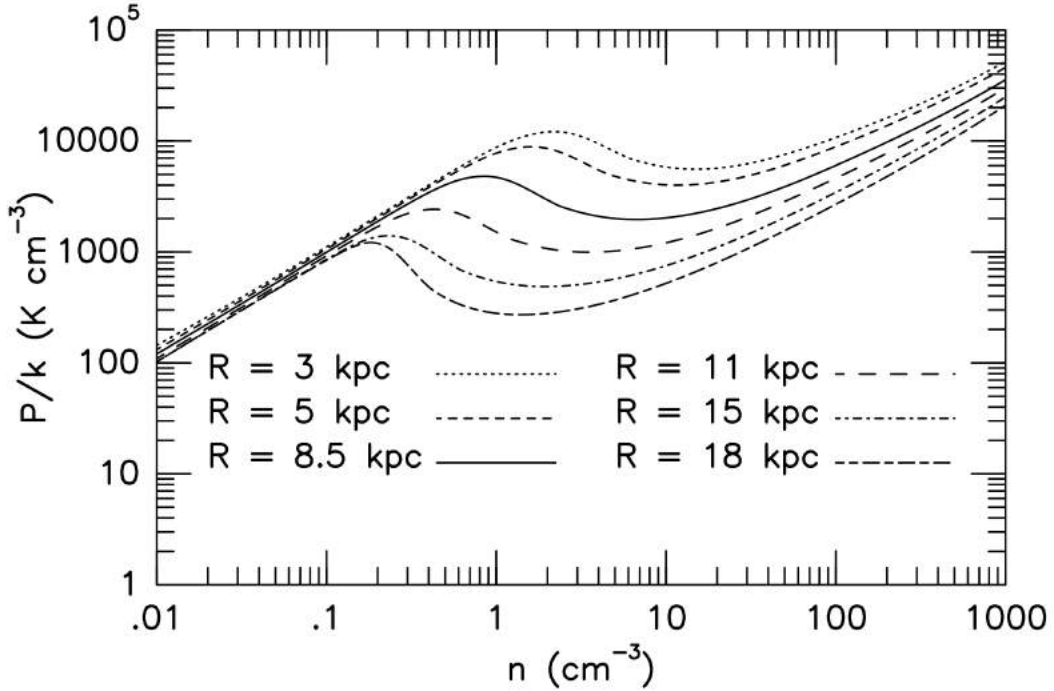


Figure 2.2: Equilibrium pressure versus hydrogen number density in the Milky Way at different distances from the centre, reprinted from Wolfire *et al.* (2003).

and the amplitude is now either decaying exponentially or growing exponentially as

$$\delta\rho = \Delta\rho e^{\pm\gamma t} e^{ikx} \quad (2.266)$$

The solution with $\exp(+\gamma t)$ grows exponentially fast without bounds: we have an *instability*. In summary, our equilibrium solution is unstable. Once the amplitude $\delta\rho$ becomes comparable to ρ_0 , we reach the non-linear stage of the instability and our linear analysis is not valid anymore. We have to solve the non-linear equations directly.

If we turn back to Figure 2.2, we see that for a typical ISM pressure of 3000 K cm^{-3} , we have a range of possible densities allowed by the equilibrium thermal model. The interval defined by $P' < 0$ is however unstable, as we have just seen. As a consequence, the ISM will naturally split into two stable phases, a dense cold molecular phase which corresponds to the rightmost stable branch where $P' > 0$, $n \simeq 100 \text{ H/cm}^3$ and $T \simeq 10 \text{ K}$, and a lower density warm atomic phase which corresponds to the leftmost stable branch where $P' > 0$, $n \simeq 0.1 \text{ H/cm}^3$ and $T \simeq 10^4 \text{ K}$. In between these two stable phases, there is a forbidden region where $P' < 0$. The thermal instability is a fundamental process that governs this phase transition within the ISM and leads to the formation of dense molecular clouds.

2.7.3 Jeans instability

We now study the propagation of sound waves in the presence of gravity. We assume an isothermal equation of state with $P = \rho a^2$. Our equilibrium state is again uniform with $\rho = \rho_0$, $v = 0$ and $P = \rho_0 a^2$. We need also the equilibrium state to satisfy Poisson equation

$$\Delta\phi = 4\pi G\rho \quad (2.267)$$

which is a problem, because the potential of an infinite uniform background is infinite. In order to restore a viable equilibrium state, we appeal to the so-called *Jeans swindle*. The trick is to modify Poisson equation into

$$\Delta\phi = 4\pi G(\rho - \rho_0) \quad (2.268)$$

so that the equilibrium solution is now $\phi = 0$ when $\rho = \rho_0$. The Jeans swindle is hard to justify rigorously. In only two cases we can motivate the extra term in the right-hand side: in the presence of rotation, the centrifugal force plays the role of an effective potential balancing gravity, and in cosmology, the expansion of the Universe cancels out exactly the mean background density. In what follows, we just assume that this modification of Poisson equation is valid. We then augment the one-dimensional Euler equations with the modified one-dimensional Poisson equation

$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial v}{\partial x} + v\frac{\partial\rho}{\partial x} = 0 \quad (2.269)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{a^2}{\rho}\frac{\partial\rho}{\partial x} = -\frac{\partial\phi}{\partial x} \quad (2.270)$$

$$\frac{\partial^2\phi}{\partial x^2} = 4\pi G(\rho - \rho_0) \quad (2.271)$$

We then consider perturbations of the form

$$\rho = \rho_0 + \delta\rho \quad (2.272)$$

$$v = 0 + \delta v \quad (2.273)$$

$$\phi = 0 + \delta\phi \quad (2.274)$$

Injecting these perturbations into the previous system of equations, we can linearize them by dropping all high order quadratic terms and obtain the following system of linear differential equations

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_0\frac{\partial}{\partial x}(\delta v) = 0 \quad (2.275)$$

$$\frac{\partial}{\partial t}(\delta v) + \frac{a^2}{\rho_0}\frac{\partial}{\partial x}(\delta\rho) = -\frac{\partial}{\partial x}(\delta\phi) \quad (2.276)$$

$$\frac{\partial^2}{\partial x^2}(\delta\phi) = 4\pi G\delta\rho \quad (2.277)$$

Here again, we take the time derivative of the mass conservation equation, swap the time and space derivative of the velocity term and inject the momentum conservation equation. We obtain the following relation

$$\frac{\partial^2}{\partial t^2}(\delta\rho) - a^2\frac{\partial^2}{\partial x^2}(\delta\rho) - \rho_0\frac{\partial^2}{\partial x^2}(\delta\phi) = 0 \quad (2.278)$$

Finally, we use the Poisson equation to replace the third term and obtain the final equation on the density perturbation

$$\frac{\partial^2}{\partial t^2}(\delta\rho) - a^2\frac{\partial^2}{\partial x^2}(\delta\rho) - 4\pi G\rho_0(\delta\rho) = 0 \quad (2.279)$$

As always, we consider solutions of the form $\delta\rho = \Delta\rho e^{i(kx - \omega t)}$ where k is real but ω can be a complex number. Injecting this in the previous differential equation and requiring $\Delta\rho \neq 0$, we obtain the following dispersion relation

$$\boxed{\omega^2 = a^2k^2 - 4\pi G\rho_0} \quad (2.280)$$

We can define a critical wave number that separates stable modes with $\omega^2 > 0$ from unstable modes with $\omega^2 < 0$.

$$k_J^2 = \frac{4\pi G\rho_0}{a^2} \quad (2.281)$$

This critical wave number can be used to define the *Jeans length* as

$$\lambda_J = \frac{2\pi}{k_J} = a\sqrt{\frac{\pi}{G\rho_0}} \quad (2.282)$$

If $k > k_J$, we have stable propagating waves, which for $k \gg k_J$ are just normal sound waves with $\frac{\omega}{k} = \pm a$.

If $k < k_J$, we have $\omega^2 < 0$ and $\omega = i\gamma$ where γ is the growth rate. This is what we call *Jeans instability*. For $k \ll k_J$, we find $\gamma = \sqrt{4\pi G\rho_0}$. This growth rate can be used to define the *free fall time* as

$$t_{\text{ff}} = \frac{1}{\gamma} = \frac{1}{\sqrt{4\pi G\rho_0}} \quad (2.283)$$

This time is the e-folding time of the growth of the amplitude of the density fluctuations. For very long wavelengths, gravity wins over pressure forces and the sound waves collapse under their own weight within one free fall time. Beyond this time, the linearisation of the equations is not valid anymore and we enter the non-linear regime of the gravitational instability. For very short wavelengths, on the other hand, pressure forces win against gravity and sound waves can propagate freely without perturbing the initial equilibrium state.

2.8 Shock waves

Another type of waves plays a fundamental role in astrophysics, the shock waves. These are not linear objects like sound waves. They are the product of the non-linear evolution of the Euler equations. Shock waves are discontinuities traveling in the fluid. Partial derivatives are not defined across discontinuities, so we must rely on particular conservation laws, called the Rankine-Hugoniot relations, to connect the two states on each side of the discontinuity. Shock waves are also dissipative. Because all flow variables are discontinuous across shocks, the Euler equations cease to be valid, as the scale of a discontinuity is obviously zero, much smaller than the collision mean free path. In fact, shock waves have a finite thickness, close to the mean free path of the particles. This thickness is due to viscous effects at small scales. The Rankine-Hugoniot relations are powerful tools. Even if we ignore these small scale viscous effects, we can still recover the correct relations connecting the upstream and downstream regions of the shock. After deriving these relations in the specific cases of adiabatic and isothermal shocks, we will apply our findings to the theory of astrophysical blast waves, used in particular to describe supernova remnants in the interstellar medium.

2.8.1 Wave steepening and shock Formation

In order to describe the process of shock formation, we need to consider the full non-linear Euler equations, with for the velocity in one dimension

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} = 0 \quad (2.284)$$

The second term is usually called the non-linear term because it features a quadratic term in the velocity perturbation. In the previous sections, this term was dropped, leading to linear differential equations. We now want to keep this term and try to solve the previous equation in

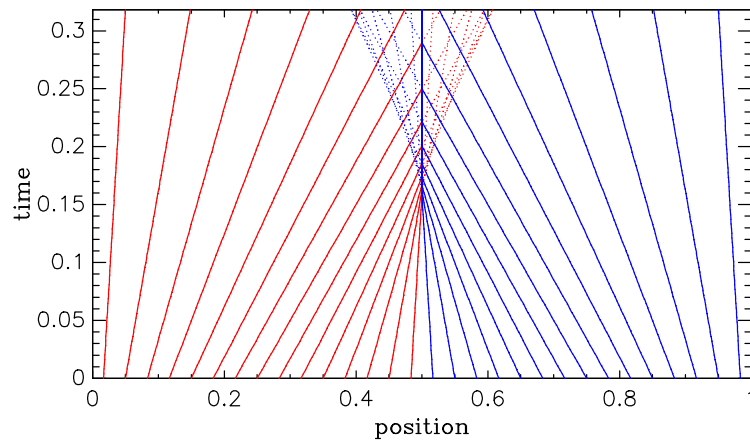


Figure 2.3: Space time diagram showing the characteristics of the Burgers equation. The dotted lines shows the trajectories after characteristics crossings. The solid lines show the regularized single valued solution and the formation of a shock front at $x = 0.5$ at time $t = 0.16$.

a slightly simplified setting. We consider here again an isothermal equation of state $P = \rho a^2$, but we restrict ourselves to the supersonic regime for which $a \ll v$. In this case, we can write

$$\left| \frac{1}{\rho} \frac{\partial P}{\partial x} \right| = \left| \frac{a^2}{\rho} \frac{\partial \rho}{\partial x} \right| \ll \left| v \frac{\partial v}{\partial x} \right| \quad (2.285)$$

and we can drop the pressure gradient and obtain the famous *Burgers equation*

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0 \quad (2.286)$$

We now want to solve this equation with the prescribed initial condition $v_0(x) = v(x, t = 0)$. For the sake of simplicity, we choose here a planar wave

$$v_0(x) = \Delta v \sin\left(\frac{2\pi}{L}x\right) \quad (2.287)$$

consistent with the Ansatz used in the previous sections. We assume without loss of generality that $\Delta v = 1$ and $L = 1$. Let us now consider fluid trajectories $x(t)$ defined by

$$\dot{x} = v(x(t), t) \quad \text{and} \quad x(0) = x_0 \quad (2.288)$$

We now define the scalar quantity

$$\alpha(t) \equiv v(x(t), t) \quad (2.289)$$

attached to each trajectory $x(t)$. Taking the time derivative and applying the chain rule gives

$$\dot{\alpha}(t) = \frac{\partial v}{\partial x} \dot{x} + \frac{\partial v}{\partial t} = 0 \quad (2.290)$$

so that $\alpha(t) = \alpha(0) = v_0(x_0)$ remains constant in time. This means that $v = v_0$ along the trajectory and the direct consequence is that trajectories are straight lines

$$x(t) = x_0 + v_0(x_0)t \quad (2.291)$$

The straight lines are called *characteristic curves*, or in short, *characteristics*.

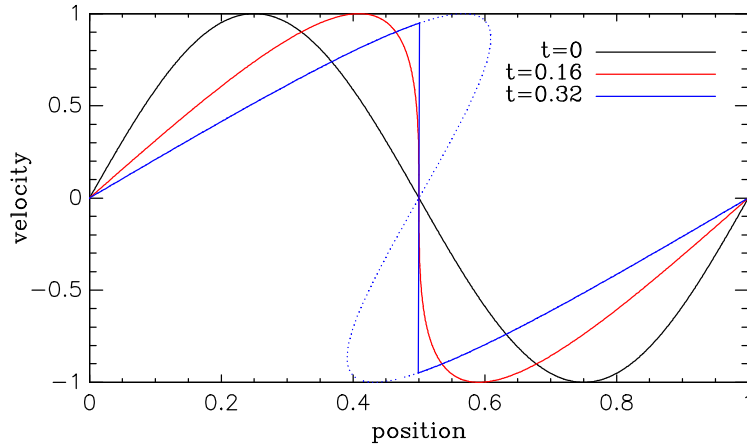


Figure 2.4: Velocity profiles of the Burgers equation at three different times. We see how the initial sine wave progressively steepens, leading to the formation of a singularity at $x = 0.5$ and $t = 0.16$, after which characteristics are crossing each other and lead to a spurious multi-valued solution (dotted lines). The correct regularized solution is shown as the discontinuous solid line and features a shock front at $x = 0.5$.

We can now write the exact solution of Burgers equation by noticing that for any point (x, t) in space-time, we can measure the velocity v and go back in time towards $t = 0$ following the characteristic $x_0 = x - vt$ and match the value of the velocity with $v_0(x_0)$. This can be written as the following implicit solution for $v(x, t)$

$$v(x, t) = v_0(x - v(x, t)t) \quad (2.292)$$

We see in Figure 2.3 that the characteristics will cross at $x = 0.5$ in a finite time. This characteristic crossing will lead to the formation of a singularity, as two different values of v will collide at the same location. Which one shall we choose? Taking the time derivative of the previous implicit solution, using again the chain rule, we find

$$\frac{\partial v}{\partial t} = \frac{\partial v_0}{\partial x} \left(-v - t \frac{\partial v}{\partial t} \right) \quad \text{so that} \quad \frac{\partial v}{\partial t} = \left[-v(x, t) \frac{\partial v_0}{\partial x} \right] / \left[1 + t \frac{\partial v_0}{\partial x} \right] \quad (2.293)$$

We see that the time derivative of v becomes singular in a finite time given by the *crossing time*

$$\frac{1}{t_{\text{cross}}} = -\frac{\partial v_0}{\partial x} \quad (2.294)$$

The minimum positive value for t_{cross} is obtained at $x = 0.5$ and corresponds to $t = 1/2\pi$. This time marks the first characteristic crossing and the formation of the shock. In Figure 2.4, we see the evolution of the velocity profile. The initial sine wave progressively steepens, as the central characteristics all converge towards $x = 0.5$. At the crossing time, the velocity profile becomes vertical and its derivative infinite. At later time, the velocity function becomes multi-valued, with 3 possible solutions at each location. We know that in this case, the validity conditions for the Euler equations are violated, as the collision mean free path will become larger than the thickness of the velocity profile. We must therefore include at least a first order description of non-LTE effects, namely viscosity. This leads to the *viscous Burgers equation*

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \nu \Delta v \quad (2.295)$$

where ν is the viscosity diffusion coefficient. Viscosity will regularize the problem and lead to a non-singular solution. In this case, the solution will become unique and single-valued

everywhere. This regularized solution is called the *viscosity solution* or the *weak solution*. Note that the viscous Burgers equation has an exact solution based on the Hopf-Cole transform (discussed in the next paragraph). The procedure is then to find the viscosity solution for a finite viscosity parameter ν and then make $\nu \rightarrow 0$, instead of setting $\nu = 0$ in the equations before solving them, which leads to an ill-defined result.

2.8.2 Hopf-Cole transformation

In this subsection, we would like to find an explicit solution to the viscous Burgers equation. Surprisingly enough, such a closed form solution exists thanks to a clever trick proposed independently by Hopf and Cole in the early 50s. We start by re-writing the one-dimensional viscous Burgers equation as

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left(\frac{v^2}{2} \right) = \frac{\partial}{\partial x} \left(\nu \frac{\partial v}{\partial x} \right) \quad (2.296)$$

We now define a new field ϕ as

$$u = -2\nu \frac{1}{\phi} \frac{\partial \phi}{\partial x} \quad (2.297)$$

that can be immediately integrated as

$$\phi(x, t) = \phi_0 \exp \left(-\frac{1}{2\nu} \int_0^x v(y, t) dy \right) \quad (2.298)$$

We can now try to replace v by ϕ in the viscous Burgers equation, computing first

$$\frac{\partial v}{\partial t} = -2\nu \left[-\frac{1}{\phi^2} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} + \frac{1}{\phi} \frac{\partial^2 \phi}{\partial t \partial x} \right] = -2\nu \frac{\partial}{\partial x} \left(\frac{1}{\phi} \frac{\partial \phi}{\partial t} \right) \quad (2.299)$$

We now move the non-linear term to the right-hand side of Burgers equation and compute

$$\frac{\partial}{\partial x} \left(\nu \frac{\partial v}{\partial x} - \frac{v^2}{2} \right) = \frac{\partial}{\partial x} \left(-2\nu^2 \left[-\frac{1}{\phi^2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} \right] - 2\nu^2 \frac{1}{\phi^2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right) \quad (2.300)$$

Owing to this clever transformation, we see that the first and the third term of this last equation cancel out. We can then finally express the viscous Burgers equation as

$$\frac{\partial}{\partial x} \left(\frac{1}{\phi} \frac{\partial \phi}{\partial t} \right) = \nu \frac{\partial}{\partial x} \left(\frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} \right) \quad (2.301)$$

We can easily integrate this differential equation in space and obtain

$$\frac{1}{\phi} \frac{\partial \phi}{\partial t} = \nu \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} + C(t) \quad (2.302)$$

We now introduce a new variable ψ defined by

$$\phi(x, t) = \psi(x, t) \exp \left(\int_0^t C(t') dt' \right) \quad (2.303)$$

We can check that

$$\frac{\partial \phi}{\partial t} = \frac{\partial \psi}{\partial t} \exp \left(\int_0^t C(t') dt' \right) + \phi C(t) \quad (2.304)$$

so that ψ now satisfies the good-old *heat equation*

$$\boxed{\frac{\partial \psi}{\partial t} = \nu \frac{\partial^2 \psi}{\partial x^2}} \quad (2.305)$$

It is easy to find the general solution of the heat equation using the convolution with the corresponding Green's function

$$G(x, t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left(-\frac{x^2}{4\nu t}\right) \quad (2.306)$$

with the initial condition defined by

$$\psi_0(x) = \psi(x, t = 0) \quad (2.307)$$

as

$$\psi(x, t) = \int_{-\infty}^{+\infty} \psi_0(y) G(x - y) dy = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \psi_0(y) \exp\left(-\frac{(x - y)^2}{4\nu t}\right) dy \quad (2.308)$$

We note in passing that

$$\psi_0(x) = \phi(x, t = 0) = \phi_0 \exp\left(-\frac{1}{2\nu} \int_0^x v_0(y) dy\right) \quad (2.309)$$

where $v_0(x)$ defines the initial conditions of our velocity field. In order to compute the final solution for the velocity field, we recall that

$$u = -2\nu \frac{1}{\phi} \frac{\partial \phi}{\partial x} = -2\nu \frac{1}{\psi} \frac{\partial \psi}{\partial x} \quad (2.310)$$

We need to compute the following derivative as

$$\frac{\partial \psi}{\partial x} = -\frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \psi_0(y) \exp\left(-\frac{(x - y)^2}{4\nu t}\right) \left(\frac{x - y}{2\nu t}\right) dy \quad (2.311)$$

to find the closed form solution of the viscous Burgers equation as

$$\boxed{v(x, t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp(-F(y)/2\nu) dy}{\int_{-\infty}^{+\infty} \exp(-F(y)/2\nu) dy} \quad \text{where} \quad F(y) = \frac{(x - y)^2}{2t} + \int_0^y v_0(z) dz} \quad (2.312)$$

2.8.3 Riemann problem

We see in the previous formula that the initial conditions are playing a central role in determining the final solution. The general case is not really useful to describe the physics of shock waves. It is actually more useful to consider a more specific set of initial conditions defined by constant values in two infinite domains, to the left and to the right of a initial discontinuity located at $x = 0$. This initial condition is called the *Riemann problem* and is a fundamental tool to design numerical methods. We thus have

$$v_0(x) = v_L \quad \text{for} \quad x < 0 \quad \text{and} \quad v_0(x) = v_R \quad \text{for} \quad x > 0. \quad (2.313)$$

We can now write the solution of the Riemann problem for the viscous Burgers equation with

$$F(y) = \frac{(x - y)^2}{2t} + v_L y \quad \text{for} \quad y < 0 \quad \text{and} \quad F(y) = \frac{(x - y)^2}{2t} + v_R y \quad \text{for} \quad y > 0 \quad (2.314)$$

which is a simple piecewise parabolic function. As we have already discussed above, we are interesting in the limit $\nu \rightarrow 0$. The Gaussian function we have in the previous integrals will

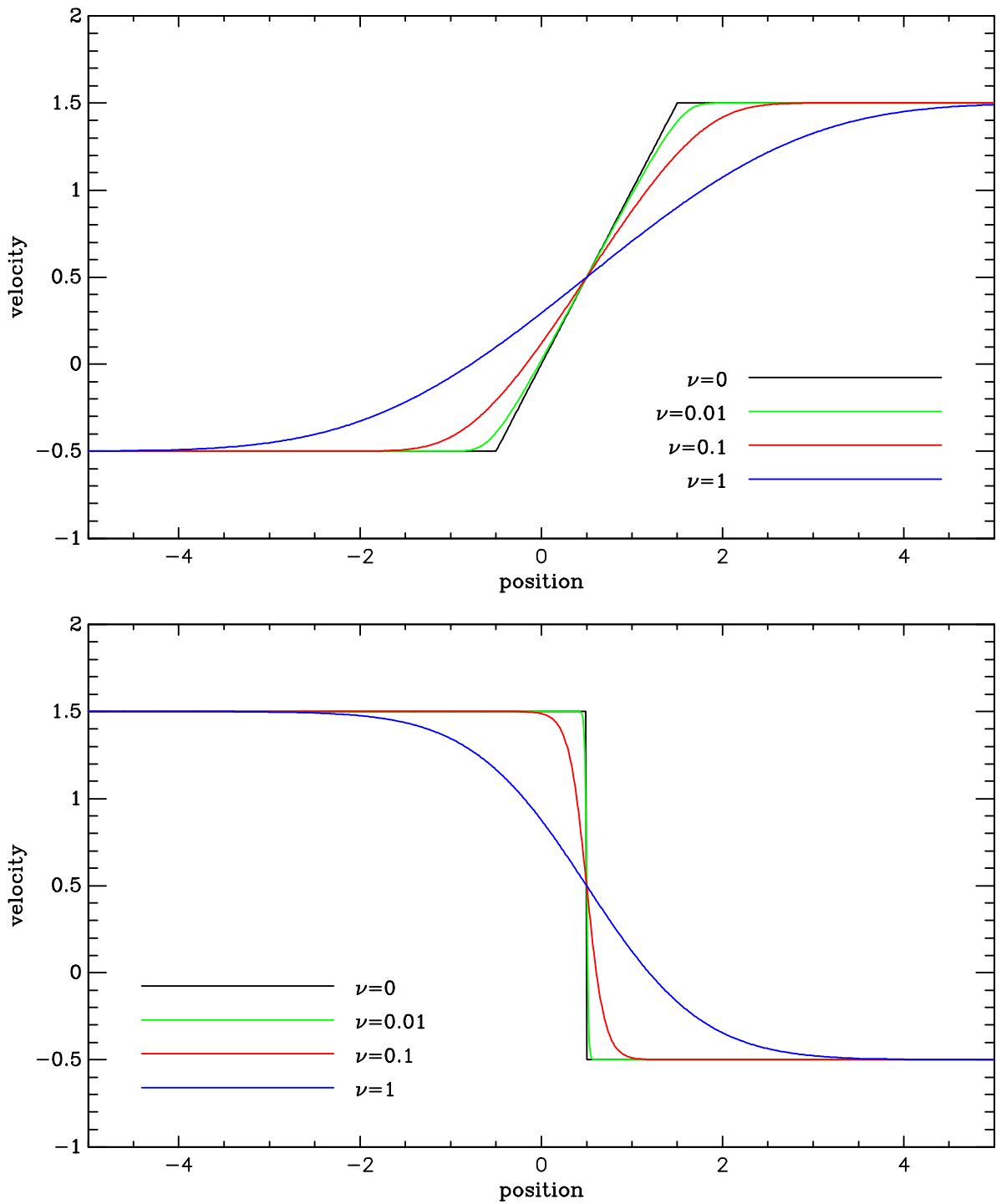


Figure 2.5: Velocity profiles of the viscous Burgers equation for a Riemann problem with a shock (bottom panel) and for a Riemann problem with a rarefaction wave (top panel) and different values for the diffusion coefficient.

converge towards the Dirac delta function, centred on the minimum of the previous (piecewise) parabola. We therefore obtain for the solution to the Riemann problem the simple result

$$v(x, t) = \frac{x - y_{\min}}{t} \quad (2.315)$$

where y_{\min} is the location of the minimum of the function F . Let's try to find the location of the minimum. In the left domain, we have find $y_{\min,L} = x - v_L t$ and in the right domain we find $y_{\min,R} = x - v_R t$. We recognise in these two equations the definitions of the left and right characteristics. The corresponding minimum values are $F_{\min,L} = v_L(x - v_L t/2)$ for the $y < 0$ and $F_{\min,R} = v_R(x - v_R t/2)$ for the $y > 0$. We have now to consider two different cases.

- $v_L > v_R$: shock wave with converging characteristics. We can define three different regions. Region 1 is defined by $x < v_R t$. In this case, the minimum is in the left domain because $y_{\min,L} < 0$ as well as $y_{\min,R} < 0$. We also have $F_{\min,L} < F_{\min,R}$. As a consequence, $y_{\min} = x - v_L t$ and $v(x, t) = v_L$. Region 2 is defined by $v_R t < x < v_L t$. In this region, both the left and right domains contains a minimum of their respective parabola. We have to find where is the global minimum by imposing $F_{\min,L} = F_{\min,R}$, which gives

$$v_L \left(x - \frac{v_L t}{2} \right) = v_R \left(x - \frac{v_R t}{2} \right) \quad \text{or} \quad x = S t \quad \text{where} \quad S = \frac{v_L + v_R}{2} \quad (2.316)$$

We have introduced S the shock speed. Indeed, we see that if $x < S t$ then the global minimum belongs to the left domain and we have $v(x, t) = v_L$ as before. On the other hand, if $x > S t$ then the global minimum is on the right and we have $y_{\min} = y_{\min,R} = x - v_R t$ so that $v(x, t) = v_R$. Finally, region 3 is defined by $x > v_L t$ so that the global minimum is clearly in the right domain and $v(x, t) = v_R$. In conclusion, the solution to the Riemann problem in this case is a shock wave solution with

$$v = v_L \quad \text{for} \quad x < S t \quad \text{and} \quad v = v_R \quad \text{for} \quad x > S t \quad (2.317)$$

- $v_L < v_R$: rarefaction wave with diverging characteristics. We can also define similarly 3 different regions. Region 1 and 3 are easy, because the minimum of the parabola is always within the corresponding left and right domain and is always the global minimum. We thus have like before for $x < v_L t$ the solution $v(x, t) = v_L$ and for $x > v_R t$ the solution $v(x, t) = v_R$. The second region in the middle is more complicated. In this case, we have $v_L t < x < v_R t$ and as a consequence the minimum of the parabola is always outside of its corresponding domain. It means that for region 2 the global minimum of $F(y)$ is always at $y_{\min} = 0$ and the solution is $v(x, t) = x/t$. In conclusion, the solution for the Riemann problem in this case is a rarefaction wave with

$$v = v_L \quad \text{for} \quad x < v_L t, \quad v = \frac{x}{t} \quad \text{for} \quad v_L t < x < v_R t, \quad v = v_R \quad \text{for} \quad x > v_R t \quad (2.318)$$

We show in Figure 2.5 the solution of the Riemann problem for the viscous Burgers equation and for various values of the diffusion coefficient ν . Interestingly, the proper inviscid solution is recovered in the limit $\nu \rightarrow 0$ and corresponds to a discontinuity in the shock case and to a inviscid rarefaction wave in the diverging characteristics case. Rarefaction waves, together with shock waves and contact discontinuities form the entire family on non-linear waves of the Euler equations. We will describe in more details contact discontinuities in a follow-up section, focusing now on the description of shock waves.

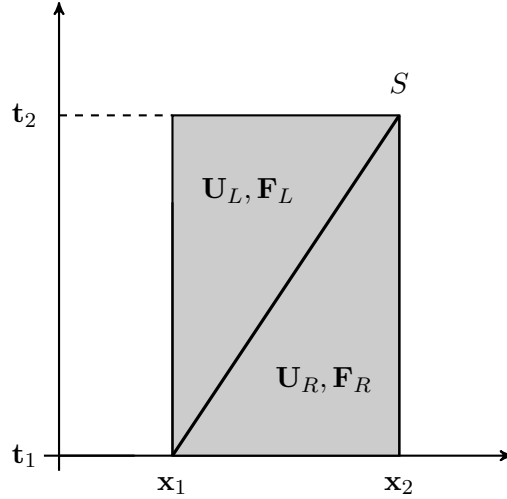


Figure 2.6: Space-time diagram with the integration region in grey

2.8.4 Rankine-Hugoniot relations

Now that we are convinced that discontinuities form systematically after a finite time in astrophysical fluid flows, we need a mathematical tool to deal with them. Indeed, the Euler equations in Lagrangian or Eulerian forms feature many spatial derivatives which are obviously not defined across discontinuities. Shock fronts have to be considered as a singular surface separating the flow in two regions: the pre-shock region that contains the unperturbed fluid and the post-shock region with the shocked material. Inside these two regions, flow variables are continuous and differentiable, but not across the shock. In order to connect these two regions across the discontinuity, we need a new set of conservation laws in *integral form*, called the Rankine-Hugoniot relations. To derive these famous relations, we write the Euler equations in the following conservative form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \quad (2.319)$$

where the vector of *conservative variables* is defined by

$$\mathbf{U} = (\rho, \rho v, E) \quad (2.320)$$

and the vector of the fluxes of the conserved variables is defined by

$$\mathbf{F} = (\rho v, \rho v^2 + P, (E + P)v) \quad (2.321)$$

These equations are one-dimensional and represent the flow evolution in the direction normal to the shock surface. Note that many other equations can be expressed in this form: the isothermal Euler equations, without the energy equation, the ideal MHD equations that include the magnetic field, the relativistic Euler equations, the traffic flow equations, the shallow water equations, etc. The Rankine-Hugoniot relations can be derived similarly for all these different system of conservation laws. We now integrate this system of equations in a square region in space-time between times t_1 and t_2 and positions x_1 and x_2 chosen close enough from each other so that the shock trajectory (or shock characteristic) can be approximated by its tangent line. The shock velocity S in the direction normal to the shock surface is therefore defined in this small region by $x_1 - x_2 = S(t_1 - t_2)$.

We now write the integral both in time and space as

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \left(\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} \right) dx dt = 0 \quad (2.322)$$

The time derivative can be integrated immediately in time and the space derivative in space, leading to

$$\int_{x_1}^{x_2} [\mathbf{U}(x, t_2) - \mathbf{U}(x, t_1)] dx + \int_{t_1}^{t_2} [\mathbf{F}(x_2, t) - \mathbf{F}(x_1, t)] dt = 0 \quad (2.323)$$

We now label the pre-shock values on the right side of the shock front as \mathbf{U}_R and \mathbf{F}_R and the post-shock region on the left as \mathbf{U}_L and \mathbf{F}_L . Since we consider a vanishingly small space-time interval, the pre-shock and post-shock values can be considered as constant both in time and space. At time $t = t_2$, one has $\mathbf{U} = \mathbf{U}_L$ everywhere, while at time $t = t_1$, $\mathbf{U} = \mathbf{U}_R$ everywhere. The first term can be written as

$$(x_2 - x_1) (\mathbf{U}_L - \mathbf{U}_R) \quad (2.324)$$

The second term can be computed noticing that at $x = x_1$, one has $\mathbf{F} = \mathbf{F}_L$ at all time, while at $x = x_2$, one has $\mathbf{F} = \mathbf{F}_R$ at all time.

$$(t_2 - t_1) (\mathbf{F}_R - \mathbf{F}_L) \quad (2.325)$$

Combining the two and using the definition of the shock speed, we finally get the Rankine-Hugoniot relations

$$\boxed{\mathbf{F}_R - \mathbf{F}_L = S(\mathbf{U}_R - \mathbf{U}_L)} \quad (2.326)$$

Note that a more convenient form can be obtained by moving from the laboratory frame to the shock frame. If the velocity field in the laboratory frame is noted v , then the shock velocity in the shock frame is given by $w = v - S$ and the Rankine-Hugoniot relations in this new frame where $S = 0$, are just $\mathbf{F}_R = \mathbf{F}_L$, much simpler indeed.

2.8.5 Burgers shocks

In the simple case of the inviscid Burgers equation, we can use Rankine-Hugoniot relation to recover the correct shock speed we have already found using the Hopf-Cole transform. Indeed, since we have $F = \frac{1}{2}v^2$ and $U = v$, we deduce immediately that

$$\frac{1}{2}v_R^2 - \frac{1}{2}v_L^2 = S(v_R - v_L) \quad \text{or} \quad S = \frac{v_L + v_R}{2} \quad (2.327)$$

in agreement with our previous result.

2.8.6 Isothermal shocks

We now apply our newly discovered relations to the case of an isothermal gas. The right region, pre-shock states are given by (ρ_R, v_R, S) , and we wish to determine the post-shock states (ρ_L, v_L) as a function of the shock speed S . In the laboratory frame, the Rankine-Hugoniot relations write

$$\rho_R v_R - \rho_L v_L = S(\rho_R - \rho_L) \quad (2.328)$$

$$\rho_R v_R^2 + \rho_R a^2 - \rho_L v_L^2 - \rho_L a^2 = S(\rho_R v_R - \rho_L v_L) \quad (2.329)$$

It is easier to move in the frame of the shock using the new velocity variable $w = v - S$. In this frame, $S = 0$ and the shock characteristic is a vertical line. The Rankine-Hugoniot relations are now much simpler

$$\rho_R w_R = \rho_L w_L \quad (2.330)$$

$$\rho_R w_R^2 + \rho_R a^2 = \rho_L w_L^2 + \rho_L a^2 \quad (2.331)$$

It is customary to define the *compression ratio* of the shock as

$$r = \frac{\rho_L}{\rho_R} \quad (2.332)$$

and the Mach number of the shock as

$$\mathcal{M} = \frac{|w_R|}{a} \quad (2.333)$$

The mass conservation relation now writes

$$w_L = \frac{1}{r} w_R \quad (2.334)$$

and the momentum conservation relation simplifies to

$$w_R^2 + a^2 = r (w_L^2 + a^2) \quad (2.335)$$

Dividing by a^2 and using the first relation, the second relation becomes

$$\mathcal{M}^2 + 1 = r \left(\frac{\mathcal{M}^2}{r^2} + 1 \right) \quad (2.336)$$

After some easy manipulations, we obtain a polynomial of degree 2 for the unknown compression ratio r

$$r^2 - r(\mathcal{M}^2 + 1) + \mathcal{M}^2 = 0 \quad (2.337)$$

Note $r = 1$ is always a solution of the Rankine-Hugoniot relations. It corresponds to a virtual shock wave moving through the fluid without affecting the flow. The physical solution is the other solution $r = \mathcal{M}^2 > 1$. One condition to get a shock wave is thus $\mathcal{M} > 1$. Moving back to the laboratory frame, we obtain the post-shock values as a function of \mathcal{M} and the pre-shock values

$$\boxed{\rho_L = \mathcal{M}^2 \rho_R} \quad (2.338)$$

$$v_L = \frac{1}{\mathcal{M}^2} (v_R - S) + S \quad (2.339)$$

The *strong shock limit* is obtained for $\mathcal{M} \rightarrow +\infty$. In this case, we see that the compression ratio becomes very large and that the post-shock velocity $v_L \simeq S$. The fluid is therefore put in motion by the shock wave and immediately follows it. Let's now consider two different examples.

- First example: Let's assume that the pre-shock velocity is zero $v_R = 0$ and the shock is moving from left to right with speed $S > a$. In this case, we have from the previous relations $\mathcal{M} = S/a > 1$. The compression ratio is fully specified by the shock speed with

$$r = \mathcal{M}^2 = \frac{S^2}{a^2} > 1 \quad (2.340)$$

and the post-shock velocity is just

$$v_L = S \left(1 - \frac{a^2}{S^2} \right) \quad (2.341)$$

- Second example: Let's assume we don't know the shock speed but we know that the fluid is initially moving from right to left with $v_R < 0$ and is crashing onto the left wall of the laboratory. This sets the boundary conditions $v_L = 0$. A shock will emerge from the

left wall with $S > 0$ and will propagate to the right, slowing down the incoming fluid. Using the previous relations, we deduce the shock speed value by solving the following polynomial of degree 2

$$S^2 - Sv_R - a^2 = 0 \quad (2.342)$$

for which the only physical solution is

$$S = \frac{v_R + \sqrt{v_R^2 + 4a^2}}{2} > a \quad (2.343)$$

The corresponding compression ratio is

$$r = 1 + \frac{v_R^2}{4a^2} \quad (2.344)$$

We see that in these two examples, the shock wave always travels supersonically with respect to the fluid at rest. The compression ratio is always greater than 1, and scales as the Mach number square, without bounds. In this isothermal case, the post-shock temperature remains constant and equal to the pre-shock temperature.

2.8.7 Adiabatic shocks

We now move to the more complex case of an ideal gas equation of state with $P = (\gamma - 1)e$. The Rankine-Hugoniot relations in the frame of the shock are

$$\rho_L w_L = \rho_R w_R \quad (2.345)$$

$$\rho_L w_L^2 + P_L = \rho_R w_R^2 + P_R \quad (2.346)$$

$$w_L(E_L + P_L) = w_R(E_R + P_R) \quad (2.347)$$

Using the definition of the total energy $E = e + \frac{1}{2}\rho w^2$ and injecting the mass conservation relation, we can write the energy equation as

$$\frac{1}{2}w_L^2 + \frac{\gamma}{\gamma - 1} \frac{P_L}{\rho_L} = \frac{1}{2}w_R^2 + \frac{\gamma}{\gamma - 1} \frac{P_R}{\rho_R} \quad (2.348)$$

In order to simplify the algebra, we consider the case where the fluid is initially at rest with $v_R = 0$ so that $w_R = -S$. We also assume that the Mach number $\mathcal{M} = S/c_s \gg 1$ with $c_s^2 = \gamma P/\rho$ the adiabatic sound speed. We also define the compression ratio as before $r = \rho_L/\rho_R$. The mass conservation relation gives

$$w_L = -\frac{S}{r} \quad (2.349)$$

The momentum conservation relation becomes

$$\rho_L w_L^2 + P_L \simeq \rho_R S^2 \quad \text{which gives} \quad \frac{P_L}{\rho_L} = \frac{S^2}{r} - \frac{S^2}{r^2} \quad (2.350)$$

where we have neglected the thermal pressure in the right-hand side owing to the strong shock condition. Finally, the energy conservation relation becomes

$$\frac{1}{2}w_L^2 + \frac{\gamma}{\gamma - 1} \frac{P_L}{\rho_L} \simeq \frac{1}{2}S^2 \quad (2.351)$$

Combining all the previous results lead to the following relation between r and S

$$\frac{1}{2} \frac{S^2}{r^2} + \frac{\gamma}{\gamma-1} \frac{S^2}{r} \frac{(r-1)}{r} = \frac{1}{2} S^2 \quad (2.352)$$

which simplifies (as for the isothermal case) into the following polynomial of degree 2

$$r^2 - \frac{2\gamma}{\gamma-1} r + \frac{\gamma+1}{\gamma-1} = 0 \quad (2.353)$$

which has the two solutions

$$r = 1 \quad \text{and} \quad r = \frac{\gamma+1}{\gamma-1} \quad (2.354)$$

We thus get the post-shock density as a function of the pre-shock density using

$$\boxed{\rho_L = \frac{\gamma+1}{\gamma-1} \rho_R} \quad (2.355)$$

The Maxwell-Boltzmann case with $\gamma = 5/3$ gives $r = 4$. The molecular equation of state with $\gamma = 7/5$ leads to $r = 6$ and relativistic fluids with $\gamma = 4/3$ have $r = 7$. Interestingly, one also recovers the isothermal limit with $\gamma \rightarrow 1$ and $r \rightarrow +\infty$. The mass conservation relation gives us the post-shock velocity with $w_L = -S/r$ and $v_L = S - S/r$ so that

$$\boxed{v_L = \frac{2}{\gamma+1} S} \quad (2.356)$$

The post-shock pressure is obtained using the momentum conservation relation with

$$\boxed{P_L = \frac{2}{\gamma+1} \rho_R S^2} \quad (2.357)$$

and the post-shock temperature is given by

$$\boxed{\frac{k_B T_L}{m} = \frac{P_L}{\rho_L} = \frac{2(\gamma-1)}{(\gamma+1)^2} S^2} \quad (2.358)$$

We see that adiabatic shocks behave very differently than isothermal shocks. In the strong shock limit, the compression ratio r is now finite and remains relatively small (from 4 to 7 depending on the equation of state). On the other hand, the post-shock temperature scales as the square of the shock speed and can become arbitrarily large, like the compression ratio in the isothermal case. In the adiabatic case, energy is conserved and the kinetic energy of the pre-shock fluid is transferred into thermal energy in the post-shock region. Shocks are thus dissipative waves which generate entropy owing to viscous effects within the shock front.

2.8.8 Radiative shocks

In the general case, shocks are neither adiabatic, nor isothermal. We need to consider cooling and heating processes in the energy equation to describe how the fluid will return to its initial temperature after the shock has passed through the fluid. Radiative processes will be described in the next chapter. They are due to inelastic collisions between atoms and molecules that emit photons. Since inelastic collisions have a probability lower than one to occur during an otherwise elastic collision, radiative processes will always be slower than viscous processes. A shock wave is thus always adiabatic. Then, slowly, radiative processes cool down the gas in the post-shock

region. After a certain distance, called the *cooling length*, the temperature goes back to the initial pre-shock temperature. Overall, for scales much larger than the cooling length, the fluid behaves as an isothermal fluid and one can use the isothermal Rankine-Hugoniot to predict the very final state.

For sake of simplicity, we consider simplified heating and cooling processes using the following energy equation

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x} (E + P) v = \rho \frac{\epsilon_0 - \epsilon}{t_{\text{cool}}} \quad (2.359)$$

where t_{cool} is the cooling time. We see that the source term in the right-hand side will force the fluid toward $\epsilon = \epsilon_0$ as the equilibrium isothermal final state. Solving the one-dimensional Euler equations in the frame of the shock leads to a stationary solution behind the shock shown in Figure 2.7. We used in this example $\gamma = 5/3$, $S = 10$, $v_R = 0$ and $\rho_R = 1$. One sees clearly a discontinuity at $x = 0$. The Rankine-Hugoniot relations we used at the discontinuity are the adiabatic ones, leading to a very small compression ratio of 4 and very large temperature jump with

$$\frac{k_B T_L}{m} = \frac{3}{16} S^2 = 18.75 \quad \text{and} \quad w_L = -\frac{1}{4} S = -2.5 \quad (2.360)$$

After this adiabatic shock, cooling starts to lower the temperature and the flow slows down and compress the gas to maintain a quasi-uniform pressure. In this example, we adopt $t_{\text{cool}} = 0.1$ so that the cooling length can be computed as $\ell_{\text{cool}} = |w_L| t_{\text{cool}} = 0.25$. One can see in Figure 2.7 that, after a couple of cooling lengths, the flow temperature returns to its initial value. The final values on the extreme left of the shock profiles are indeed exactly the predicted jumps of the isothermal Rankine-Hugoniot relations. If the relevant scales of the problem are much larger than the cooling length, one can ignore the structure of the radiative shock and use directly the isothermal shock predictions.

2.9 Astrophysical blast waves

We now apply our recent knowledge on shock waves to one of the most famous fluid dynamics problem in astrophysics, namely the propagation of a spherical blast wave in the interstellar medium. Supernova explosions are typical examples of such explosive events. The progenitor star is violently expanding in its immediate environment and the ejecta are slowed down by the surrounding gas. As a consequence, a shock wave is launched very rapidly and propagates into the ISM supposed to be at rest. In this early phase, radiative losses are negligible and we can use the ideal gas equation of state with $P = (\gamma - 1)e$. The key point of the theory of spherical blast wave is that there are no preferred length scale involved: the explosion can be seen as a point explosion and a scale-free spherical flow develops around it. The energy is so large that we can safely consider the strong shock limit with $M \gg 1$. The only parameters are the initial gas density ρ_0 and the explosion energy E_0 . For supernovae, the explosion energy is close to 10^{51} erg, and the typical ISM density varies between 0.1 and 100 H/cm³. The spherical shock radius is $R(t)$, only depending on time, with $R(0) = 0$. The shock velocity is defined by

$$S = \dot{R} \quad (2.361)$$

Using the adiabatic Rankine-Hugoniot relations, in the laboratory frame and along the normal to the shock surface, we get the following post-shock quantities immediately after the discontinuity

$$\rho_S = \frac{\gamma + 1}{\gamma - 1} \rho_0, \quad v_S = \frac{2}{\gamma + 1} S, \quad P_S = \frac{2}{\gamma + 1} \rho_0 S^2 \quad (2.362)$$

We still need to find the time evolution of the shock radius and the profiles of the various physical quantities after the shock front.

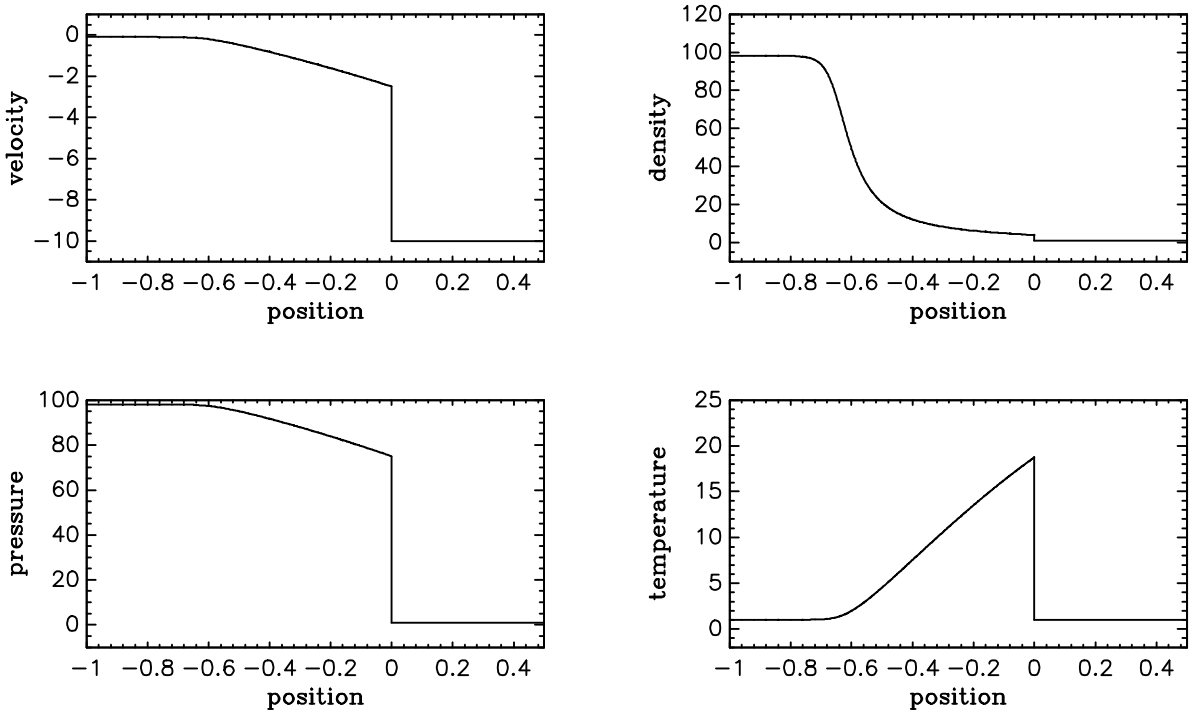


Figure 2.7: Flow solution for a radiative shock wave. The adiabatic shock front is located at $x = 0$, followed by a radiative layer of thickness equal to the cooling length. After the radiative layer, the flow variables finally reach the predicted value from the isothermal Rankine-Hugoniot relations.

2.9.1 Spherical uniform model

As a first attempt to solve the problem, we assume that the shock heated gas is uniformly distributed within the shock radius, and that the density, velocity and thermal pressure are all equal to their post-shock values. We can compute the kinetic energy within the sphere as

$$E_{\text{kin}} = \int_0^{R(t)} \frac{1}{2} \rho_S v_S^2 4\pi r^2 dr = \frac{8\pi/3}{(\gamma+1)(\gamma-1)} \rho_0 S^2 R^3 \quad (2.363)$$

and the total internal energy as

$$E_{\text{therm}} = \int_0^{R(t)} \frac{P_S}{\gamma-1} 4\pi r^2 dr = \frac{8\pi/3}{(\gamma+1)(\gamma-1)} \rho_0 S^2 R^3 \quad (2.364)$$

Interestingly, the total energy is distributed into exactly half kinetic and half thermal energy. Requiring that the total energy is conserved and equal to the initial explosion energy, we find the following differential equation for $R(t)$

$$\frac{16\pi/3}{(\gamma+1)(\gamma-1)} \rho_0 \dot{R}^2 R^3 = E_0 \quad \text{with the solution} \quad R(t) \simeq \left(\frac{E_0}{\rho_0} \right)^{1/5} t^{2/5} \quad (2.365)$$

where we used $\gamma = 5/3$. This is the famous *energy-conserving* Sedov solution, that the Russian physicist Leonid Sedov found during World War II. We can deduce the corresponding shock speed as

$$S = \dot{R} = \frac{2}{5} \left(\frac{E_0}{\rho_0} \right)^{1/5} t^{-3/5} \quad (2.366)$$

The total scalar momentum of the blast wave is given by the product of the mass and the uniform fluid radial velocity

$$p_S = \frac{4\pi}{3} R^3 \rho_S v_S = \frac{8\pi}{3(\gamma-1)} \left(\frac{E_0}{\rho_0} \right)^{4/5} t^{3/5} \quad (2.367)$$

We see that, although the shock velocity is slowly decreasing with time, the radial momentum increases with time as the shock radius expands in the ISM. The Sedov solution is valid until radiative effects become important, and the shock evolves into the isothermal regime.

2.9.2 Self-similar Sedov solution

The uniform blast wave solution we have found for $R(t)$ is just a simple power law. Together with the fact that there is no preferred absolute length scale in the problem, and that the Euler equations are scale-free, it is very tempting to postulate that the density, velocity and pressure profiles within the blast wave are *self-similar*. The self-similar solutions are defined by $\rho(r, t) = \rho_S \tilde{\rho}(x)$, $v(r, t) = v_S \tilde{v}(x)$ and $P(r, t) = P_S \tilde{P}(x)$, with $x = \frac{r}{R(t)}$ the self-similar coordinate. The shock location is not necessarily at $x = 1$ because our profiles will not be uniform and we won't find exactly the previous solution. We define the shock radius by x_S . At the shock radius, we have $\tilde{\rho}(x_S) = 1$, $\tilde{v}(x_S) = 1$ and $\tilde{P}(x_S) = 1$. In words, one says that the profiles remain similar to themselves (self-similar) after an appropriate scaling of the flow variables.

We will now derive new differential equations of the single self-similar variable x instead of two space-time coordinates (r, t) . The 1D spherical Euler equations write

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0 \quad (2.368)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} = 0 \quad (2.369)$$

We write the energy equation using the internal energy density and replace e by the pressure using $P = (\gamma - 1)e$.

$$\frac{\partial P}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 P v) + (\gamma - 1) P \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v) = 0 \quad (2.370)$$

We will now change variables and replace all spatial derivatives by

$$\frac{\partial(\cdot)}{\partial r} = \frac{1}{R(t)} \frac{\partial(\cdot)}{\partial x} \quad (2.371)$$

The time derivative is more tricky because x depends on time through $R(t)$. For example, one has to apply the chain rule and obtain

$$\frac{\partial \rho}{\partial t} = \rho_S \tilde{\rho}'(x) \left(-\frac{r}{R^2} \dot{R} \right) = -\left(\rho_S \frac{\dot{R}}{R} \right) (\tilde{\rho}' x) \quad (2.372)$$

where we used the notation

$$\tilde{\rho}'(x) = \frac{\partial \tilde{\rho}}{\partial x} \quad (2.373)$$

We then compute the divergence term in the mass conservation equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = \left(\frac{\rho_S v_S}{R} \right) \frac{1}{x^2} \frac{\partial}{\partial x} (x^2 \tilde{\rho} \tilde{v}) \quad (2.374)$$

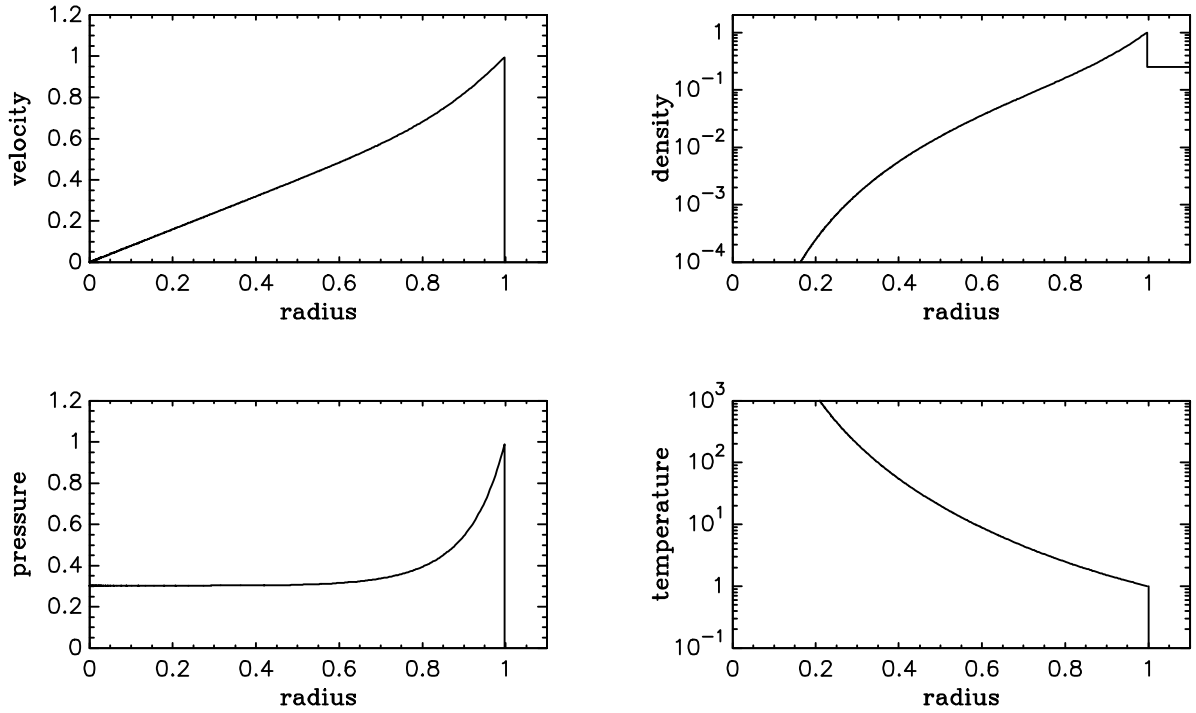


Figure 2.8: Sedov solution: spherical profiles of the scaled flow variables, normalized to 1 at the shock location $x_S = 1$.

and combine it with the time derivative to get

$$-\left(\rho_S \frac{\dot{R}}{R}\right) (\tilde{\rho}' x) + \left(\frac{\rho_S v_S}{R}\right) \frac{1}{x^2} \frac{\partial}{\partial x} (x^2 \tilde{\rho} \tilde{v}) = 0 \quad (2.375)$$

Injecting the value of the post-shock velocity

$$v_S = \frac{2}{\gamma + 1} \dot{R} \quad (2.376)$$

we see that all scaling variables cancel out and we are left with a dimensionless equation

$$-\tilde{\rho}' x + \frac{2}{(\gamma + 1)} \frac{1}{x^2} \frac{\partial}{\partial x} (x^2 \tilde{\rho} \tilde{v}) = 0 \quad (2.377)$$

Developing the divergence term, we obtain our first differential equation

$$\boxed{\tilde{\rho}' \left(\frac{2}{\gamma + 1} \tilde{v} - x \right) + \tilde{v}' \left(\frac{2}{\gamma + 1} \right) \tilde{\rho} = -\frac{4}{\gamma + 1} \frac{\tilde{\rho} \tilde{v}}{x}} \quad (2.378)$$

We now turn to the momentum equation, which writes in spherical coordinates

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} = 0 \quad (2.379)$$

We have to be careful because we use $v = v_S \tilde{v}(x)$ and the scaling variable v_S now depends on time. The time derivative thus writes

$$\frac{\partial v}{\partial t} = \dot{v}_S \tilde{v} + v_S \tilde{v}' \left(-x \frac{\dot{R}}{R} \right) = \frac{2}{\gamma + 1} \dot{R} \tilde{v} - \frac{2}{\gamma + 1} \frac{\dot{R}^2}{R} (\tilde{v}' x) \quad (2.380)$$

The spatial derivatives can then be computed. First, the pressure gradient,

$$\frac{1}{\rho} \frac{\partial P}{\partial r} = \frac{P_S}{\rho_S R} \frac{\tilde{P}'}{\tilde{\rho}} = \frac{2(\gamma-1)}{(\gamma+1)^2} \frac{\dot{R}^2}{R} \frac{\tilde{P}'}{\tilde{\rho}} \quad (2.381)$$

and, second, the non-linear term

$$v \frac{\partial v}{\partial r} = \frac{v_S^2}{R} \tilde{v} \tilde{v}' = \frac{4}{(\gamma+1)^2} \frac{\dot{R}^2}{R} \tilde{v} \tilde{v}' \quad (2.382)$$

Since $v_S \propto t^{-3/5}$, we have

$$\ddot{R} = -\frac{3}{2} \frac{\dot{R}^2}{R} \quad (2.383)$$

which makes all the scaling variables cancel out. We can finally collect all the terms and obtain our second differential equation.

$$\boxed{\tilde{v}' \left(\frac{2}{\gamma+1} \tilde{v} - x \right) + \tilde{P}' \left(\frac{\gamma-1}{\gamma+1} \right) \frac{1}{\tilde{\rho}} = \frac{3}{2} \tilde{v}} \quad (2.384)$$

We can now repeat the same procedure for the energy equation (not shown here, left as an exercise for the reader) and obtain our third differential equation

$$\boxed{\tilde{P}' \left(\frac{2}{\gamma+1} \tilde{v} - x \right) + \tilde{v}' \left(\frac{2\gamma}{\gamma+1} \right) \tilde{P} = 3\tilde{P} - \frac{4\gamma}{\gamma+1} \frac{\tilde{P}\tilde{v}}{x}} \quad (2.385)$$

Surprisingly, this system of differential equations has an exact solution, but the derivation is quite tedious. An easier approach is to solve it numerically using any integration technique, starting at $x_S = 1$, and solving numerically the previous system with initial conditions $\tilde{\rho} = \tilde{v} = \tilde{P} = 1$ until one reaches $x = 0$ at the centre of the sphere. The numerical solution is plotted in Figure 2.8. We see that the velocity profile is quasi-linear with respect to the radius, a much better approximation than a constant value. The pressure, on the other hand, is more uniform. The density and the temperature, however, are clearly not uniform. They rise and sink steeply towards the origin, leading to the formation of a very hot, underdense bubble. Since the profiles are far from our first uniform guess, the total energy will not add up to 1. The shock radius has to be adjusted so that the total energy of the explosion is strictly conserved by the solution. Integrating the numerical profiles from 0 to 1 leads to a total energy of $E_{\text{tot}} \simeq 0.4936$ in self-similar units. Since the total energy scales as R_S^5 , we deduce that the exact shock location is at $x_S = E_{\text{tot}}^{-1/5} \simeq 1.1517$

2.9.3 Momentum-conserving shell

When cooling processes start to become effective, the assumption of strict energy conservation ceases to be valid. We can estimate when this happens by computing the total cooling rate of the Sedov solution as

$$\dot{E}_{\text{cool}} = \int_0^{R(t)} n_H^2 \Lambda(T) 4\pi r^2 dr \quad (2.386)$$

and then, in a second step, the total energy radiated since the time of the explosion. The cooling time t_{cool} is defined as the time we need to radiate away the initial explosion energy

$$\int_0^{t_{\text{cool}}} \dot{E}_{\text{cool}} dt = E_0 \quad (2.387)$$

At that time, the supernova remnant has reached the cooling radius defined by

$$R_{\text{cool}} = R_S(t_{\text{cool}}) \quad (2.388)$$

Beyond this radius, the flow will quickly become isothermal. In the strong shock limit, the Rankine-Hugoniot relation will give us

$$\rho_S \simeq \mathcal{M}^2 \rho_0 = \frac{S^2}{a^2} \rho_0 \quad \text{and} \quad v_S \simeq S \quad (2.389)$$

We see that, as in the isothermal shock case, the fluid after the shock immediately follows the shock front. The ISM will therefore accumulate in a very thin shell right behind the shock. The total mass swept up by the shock is equal to the mass of ISM enclosed by the shock surface

$$M_S = \frac{4\pi}{3} \rho_0 R^3(t) \quad (2.390)$$

and is entirely located in a thin cold shell. The rest of the sphere is filled with the remnant of the Sedov phase, namely a bubble of low density hot gas. This hot gas is called the *hot interstellar medium* and represents the third main phase of the ISM (after the warm neutral medium and the cold molecular gas discussed in the thermal instability section). Obviously, during this isothermal evolution, the energy is not conserved. The shock is piling up gas like a snow-plow piling up snow on a piste and this thin shell conserves its momentum. The initial condition of the snow-plow phase is the final state of the Sedov phase, so that the conserved momentum is

$$p_{\text{cool}} = p_S(t_{\text{cool}}) \quad (2.391)$$

For $t > t_{\text{cool}}$, we write that the momentum is constant as

$$M_S \dot{R} = p_{\text{cool}} \quad (2.392)$$

This gives us a new differential equation for the shock radius

$$R^3 \dot{R} = \frac{3p_{\text{cool}}}{4\pi\rho_0} \quad (2.393)$$

Integrating this gives us a new solution for the time evolution of the shock radius.

$$R(t) = \left(\frac{3p_{\text{cool}}}{\pi\rho_0} \right)^{1/4} t^{1/4} \simeq \left(\frac{p_{\text{cool}}}{\rho_0} \right)^{1/4} t^{1/4} \quad (2.394)$$

We see that in this case, the shock is moving much slower. At some point, the shock velocity will slow down so much that the shock front will become subsonic and the bubble will stop expanding. The final stage is therefore a hot bubble surrounded by a dense shell of cold gas, in which new stars will form and after some time explode in supernovae, and the whole cycle will repeat.

2.10 Astrophysical turbulence

We now turn to one of the most important topics in modern theory of astrophysical fluids, namely the theory of turbulence. It plays a fundamental role in providing new mass, momentum and energy transport phenomena. In engineering and climate modeling, the impact of turbulence is major, and various fluid instabilities that seed turbulence are responsible for many important processes like the theory of flight, boundary layers, global energy transfer in the oceans or in the atmosphere, to name a few. In astrophysics, turbulence plays a fundamental role in the interstellar medium and probably determines how stars acquire their mass. We have already seen that turbulence controls the accretion rate in disks. It is also responsible for a very important energy transfer mechanism in stars and planets called convection.

2.10.1 Some fundamental fluid properties

Before we describe in greater details the theory of turbulence, we need to explain a few basic properties of fluid flows that we will appeal to in the following section. It is therefore very important to become familiar with these concepts, as they are the building blocks of what follows.

Adiabatic evolution

In absence of cooling and heating processes, and far away from shocks, the Euler equations are said to be *adiabatic*. In other words, as we will prove now, the entropy is strictly conserved along trajectories. If, on top of that, the initial entropy is homogeneous to start with, then it will remain homogeneous at all time. The flow is said to be *isentropic*. It is important to understand that adiabatic is not equivalent to isentropic. The latter is a stronger condition on the flow than the former. Let's write the Euler equations in Lagrangian form.

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v}, \quad \rho \frac{D\epsilon}{Dt} = -P\nabla \cdot \mathbf{v}, \quad \frac{1}{\epsilon} \frac{D\epsilon}{Dt} = -(\gamma - 1)\nabla \cdot \mathbf{v} \quad (2.395)$$

where in the rightmost equation, we have used the equation of state of an ideal gas $P = (\gamma - 1)\rho\epsilon$. Combining the mass and energy conservation equations gives the following relation between the mass density and the specific energy

$$\frac{1}{\epsilon} \frac{D\epsilon}{Dt} = (\gamma - 1) \frac{1}{\rho} \frac{D\rho}{Dt} \quad (2.396)$$

We define the *pseudo-entropy* (or the entropy for short), which is not the same as the true entropy defined in the chapter on kinetic theory, as

$$S \equiv \frac{P}{\rho^\gamma} = (\gamma - 1) \frac{\epsilon}{\rho^{\gamma-1}} \quad (2.397)$$

We compute the logarithmic Lagrange derivative of the entropy

$$\frac{1}{S} \frac{DS}{Dt} = \frac{1}{\epsilon} \frac{D\epsilon}{Dt} - (\gamma - 1) \frac{1}{\rho} \frac{D\rho}{Dt} = 0 \quad (2.398)$$

In conclusion, the adiabatic evolution of the fluid, in absence of cooling and heating, and far away from shocks, is described by a very simple equation, namely

$$\boxed{\frac{DS}{Dt} = 0} \quad (2.399)$$

This simple equation just states that the entropy is conserved along the trajectory of each fluid element. If \mathbf{x}_i is the initial location of the fluid element at time t_i , and \mathbf{x}_f its final position at time t_f , then one has for an adiabatic evolution $S_f = S_i$ or

$$\frac{P(\mathbf{x}_f, t_f)}{\rho(\mathbf{x}_f, t_f)^\gamma} = \frac{P(\mathbf{x}_i, t_i)}{\rho(\mathbf{x}_i, t_i)^\gamma} \quad (2.400)$$

In contrast, isentropic flows have $S(\mathbf{x}, t) = S_0$ everywhere, or

$$P(\mathbf{x}, t) = S_0 \rho^\gamma(\mathbf{x}, t) \quad (2.401)$$

We see that isentropic flows are a particular case of a polytropic equation of state for which $P = A\rho^\Gamma$ with $A = S_0$ and $\Gamma = \gamma$. Note that in general the polytropic exponent Γ comes from external processes, such as cooling or heating, while the ideal gas adiabatic exponent γ comes from microscopic processes describing the collisions.

Vorticity

A very important quantity used to describe turbulence is the vorticity. Vortices are fluid elements for which the vorticity, defined as $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, is constant and follows the displacements of the fluid element. Vortices are also called *eddies*. Examples of such vortices are cyclones in the atmosphere of the tropical regions. We now derive the famous *vorticity equation*, using the momentum conservation equation, including viscosity, as

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P - \nabla\phi + \nu\Delta\mathbf{v} \quad (2.402)$$

with $\nu = \mu/\rho = \text{cst}$. The non-linear term, the one responsible for shock formation, can be decomposed into

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \left(\frac{v^2}{2} \right) + (\nabla \times \mathbf{v}) \times \mathbf{v} \quad (2.403)$$

which, after introducing the vorticity $\nabla \times \mathbf{v} = \boldsymbol{\omega}$ gives

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{v^2}{2} \right) + \boldsymbol{\omega} \times \mathbf{v} = -\frac{1}{\rho}\nabla P - \nabla\phi + \nu\Delta\mathbf{v} \quad (2.404)$$

We now take the curl of the previous equation, using the two following vector relations: $\nabla \times (\nabla\phi) = 0$ and $\nabla \times \Delta\mathbf{v} = \Delta(\nabla \times \mathbf{v})$, we obtain the vorticity equation

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = \frac{1}{\rho^2}\nabla\rho \times \nabla P + \nu\Delta\boldsymbol{\omega}} \quad (2.405)$$

This equation is among the most important results in fluid dynamics. It contains three terms governing the time evolution of the vorticity. The first term (that appears as the second term on the left-hand side of the equation) corresponds to the transport of vorticity. Although it is not strictly a Lagrange derivative, it still means that, aside from the effect of *vortex-stretching*, the eddies are basically following the Lagrangian flow. The second important term is the first on the right-hand side of the equation: this is the so called *baroclinic term*, which creates vorticity due to the interaction between pressure and density gradients. This term is in most circumstances zero. For example, an incompressible fluid has $\rho = \rho_0$, so $\nabla\rho = 0$ and the baroclinic term vanishes. An isothermal fluid or a fluid with a polytropic equation of state will have

$$\nabla P = P'(\rho)\nabla\rho \quad (2.406)$$

so that the pressure gradient is always colinear to the density gradient and the curl in the baroclinic term will also vanish. In general, in order to create vorticity, one need to have misaligned pressure and density gradients. This can happen if certain heating terms are present in a localised region, or if two shock waves interact and create a triple point. These situations for which vorticity is created through the baroclinic term are usual associated to fluid instabilities. The third important term is the last one on the right-hand side: viscous dissipation. One recognizes the heat equation, this one for the vorticity. Since the molecular diffusion coefficient is very small $\nu = \lambda_{\text{coll}} a$, the dissipation of eddies will take a long time, and they can be relatively long-lived features in the flow. A strong consequence of this equation is that if the baroclinic term is zero and no other source of vorticity are present, then vorticity will decay and remain zero for ever. We have a *curl-free flow* and it is justified to use the simple version of the second Bernoulli theorem.

Helmholtz decomposition theorem

This theorem is also known as the fundamental theorem of vector calculus. It states that any vector field \mathbf{v} can be decomposed into the sum of the gradient of a scalar potential and the curl of a vector potential

$$\mathbf{v} = \nabla\varphi + \nabla \times \mathbf{A} \quad (2.407)$$

This decomposition is unique and both potentials can be found by solving two different Poisson equations

$$\Delta\varphi = \nabla \cdot \mathbf{v} = \theta \quad (2.408)$$

$$\Delta\mathbf{A} = \nabla \times \mathbf{v} = \boldsymbol{\omega} \quad (2.409)$$

with the proper boundary conditions. We didn't really simplify the problem, since now we have one vector field and one scalar field instead of just one vector field. The added value of the Helmholtz decomposition will become clear now. Let's consider two examples.

- Let us consider a curl-free flow, for which $\boldsymbol{\omega} = 0$. We also assume the boundary condition at infinity is set to zero. Then, the corresponding vector potential is $\mathbf{A} = 0$ and we have a pure *potential flow* with $\mathbf{v} = \nabla\varphi$.
- Let us now consider in addition that our flow is incompressible, so that $\theta = 0$. Using the mass conservation equation, we see that

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{v} = -\theta \quad (2.410)$$

so that $\theta = 0$ is equivalent to $\rho = \rho_0$. In our study of sound waves, we have also seen that the amplitude of the density perturbation is related to the amplitude of the velocity perturbation by

$$\frac{\Delta\rho}{\rho_0} = \pm \frac{\Delta v}{a} \quad (2.411)$$

We conclude that incompressible flows are also subsonic flows for which $v \ll a$. In this case, since $\theta = 0$, we have a potential flow for which the scalar potential satisfies Laplace equation: $\Delta\varphi = 0$.

Bernoulli theorem revisited

Since our incompressible flow is also a potential flow, we can go back to the second Bernoulli theorem, which was written as

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{v^2}{2} + \phi + \Pi \right) = \mathbf{v} \times \boldsymbol{\omega} \quad (2.412)$$

and simplify it because with $\Pi = P/\rho_0$, $\boldsymbol{\omega} = 0$ and $\mathbf{v} = \nabla\phi$. This gives us

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \phi + \frac{P}{\rho_0} \right) = 0 \quad (2.413)$$

If the gradient of a scalar quantity is zero everywhere, it means the quantity is uniform, so we obtain

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \phi + \frac{P}{\rho_0} = C(t)} \quad (2.414)$$

where the Bernoulli constant $C(t)$ is the same everywhere in the fluid but possibly depends on time. This form of the Bernoulli theorem will prove very useful in what comes next.

Contact discontinuity and the kinematic condition

We now describe a new type of discontinuity which is not a shock but the interface between two fluids. This discontinuity is therefore not a wave that propagates through the fluid at a different velocity S but a material interface. The key point in describing a contact discontinuity is the requirement that the two fluids and the interface remain in contact. In the direction perpendicular to the interface, we can write the Rankine-Hugoniot relations for a general discontinuity as

$$\rho_1 v_1 - \rho_2 v_2 = S(\rho_1 - \rho_2) \quad (2.415)$$

$$(\rho_1 v_1^2 + P_1) - (\rho_2 v_2^2 + P_2) = S(\rho_1 v_1 - \rho_2 v_2) \quad (2.416)$$

$$(E_1 + P_1) v_1 - (E_2 + P_2) v_2 = S(E_1 - E_2) \quad (2.417)$$

where we have defined the normal component of the velocities using the unit vector normal to the interface \mathbf{n} as

$$v_1 = \mathbf{v}_1 \cdot \mathbf{n} \quad \text{and} \quad v_2 = \mathbf{v}_2 \cdot \mathbf{n} \quad \text{and} \quad S = \mathbf{v}_1 \cdot \mathbf{n} \quad (2.418)$$

The *contact condition*, also called the “no vacuum” condition, is just

$$v_1 = v_2 = S \quad (2.419)$$

We see that the mass conservation relation is trivially satisfied. On the other hand, the momentum and total energy conservation relations require also pressure equality across the interface as

$$P_1 = P_2 \quad (2.420)$$

These two conditions are the Rankine-Hugoniot relations for contact discontinuities. Note that contact discontinuities have in general a complex shape. In what follows, we restrict ourselves to two dimensional flows, but it can be easily generalized to 3D. We will now derive what is called *the kinematic condition on the interface*. We are in a Cartesian coordinate system with

coordinates (x, z) . If we represent the shape of the interface at any given time by the geometrical function $z = \eta(x, t)$, we can define the interface position vector and its velocity by

$$\mathbf{r}_I = (x(t), \eta(x, t)) \quad (2.421)$$

$$\mathbf{v}_I = \left(\dot{x}, \frac{\partial \eta}{\partial x} \dot{x} + \frac{\partial \eta}{\partial t} \right) \quad (2.422)$$

In the second equation, we have used the chain rule on the function η . The “no-vacuum” condition writes $\mathbf{v}_I \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$. Note for a contact discontinuity, we can authorize slipping motions so that $\mathbf{v}_I \cdot \mathbf{t} \neq \mathbf{v} \cdot \mathbf{t}$, where \mathbf{t} is the tangent vector to the interface. We can compute easily the unit tangent vector and the unit normal vector as

$$\mathbf{t} = \left(1, \frac{\partial \eta}{\partial x} \right) / \sqrt{1 + \left(\frac{\partial \eta}{\partial x} \right)^2} \quad \text{and} \quad \mathbf{n} = \left(-\frac{\partial \eta}{\partial x}, 1 \right) / \sqrt{1 + \left(\frac{\partial \eta}{\partial x} \right)^2} \quad (2.423)$$

We now impose the contact condition for each fluid as $\mathbf{v} \cdot \mathbf{n} = \mathbf{v}_I \cdot \mathbf{n}$ which writes

$$-v_x \frac{\partial \eta}{\partial x} + v_z = -\dot{x} \frac{\partial \eta}{\partial x} + \dot{x} \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} \quad (2.424)$$

which simplifies into the final kinematic condition on the interface

$$\boxed{\frac{\partial \eta}{\partial t} + v_x \frac{\partial \eta}{\partial x} = v_z} \quad (2.425)$$

2.10.2 Rayleigh-Taylor instability

We consider two fluids with constant densities ρ_1 and ρ_2 , separated by an interface initially strictly horizontal. Fluid 1 is initially below $z = 0$ and fluid 2 is above $z = 0$. The gravity field is constant and points downwards.

$$\mathbf{g} = -g\mathbf{e}_z \quad \text{with} \quad g > 0 \quad (2.426)$$

Our initial state is described by the hydrostatic equilibrium equation with

$$-\frac{1}{\rho} \frac{\partial P}{\partial z} = g \quad \text{and} \quad \mathbf{v} = 0 \quad (2.427)$$

Integrating this equation in fluid 1 and 2 gives for the equilibrium pressure profile

$$P_1 = P_0 - \rho_1 g z \quad \text{and} \quad P_2 = P_0 - \rho_2 g z \quad (2.428)$$

We see that our two fluids meet at $z = 0$ with $P_1 = P_2 = P_0$.

Perturbations

Now that our equilibrium state has been specified, we consider small perturbations with velocities $v_1 \ll a$ and $v_2 \ll a$ so that, at least in the beginning, the flow will remain subsonic and curl-free. We thus have two potential flows with

$$\mathbf{v}_1 = \nabla \varphi_1, \quad \Delta \varphi_1 = 0 \quad \text{and} \quad \mathbf{v}_2 = \nabla \varphi_2, \quad \Delta \varphi_2 = 0 \quad (2.429)$$

We use for our scalar potential a general solution of the form

$$\varphi_1(x, z, t) = \varphi_1(z) e^{i(kx - \omega t)} \quad \text{and} \quad \varphi_2(x, z, t) = \varphi_2(z) e^{i(kx - \omega t)} \quad (2.430)$$

where both $\varphi_1(z)$ and $\varphi_2(z)$ will be determined shortly. For the interface, we use a planar wave solution of the form

$$\eta(x, t) = \Delta\eta e^{i(kx - \omega t)} \quad (2.431)$$

where $\Delta\eta$ is the fixed initial amplitude of the wave. Let's first solve our two Laplace equations in the two domains. For fluid 1, we have

$$\Delta\varphi_1 = \frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial z^2} = -k^2 \varphi_1(z) e^{i(kx - \omega t)} + \varphi_1(z)'' e^{i(kx - \omega t)} = 0 \quad (2.432)$$

Dividing out the exponentials gives us the following differential equation

$$\varphi_1(z)'' = k^2 \varphi_1(z) \quad (2.433)$$

for which the general solution is

$$\varphi_1(z) = \Delta\varphi_1^+ e^{+kz} + \Delta\varphi_1^- e^{-kz} \quad (2.434)$$

where $\Delta\varphi_1^+$ and $\Delta\varphi_1^-$ are fixed amplitude set by the initial conditions. Since fluid 1 is defined in the entire domain $z < 0$, we need to set $\Delta\varphi_1^- = 0$ to avoid a divergence at $z \rightarrow -\infty$. In conclusion, the final form for our solution for φ_1 reads

$$\varphi_1(x, z, t) = \Delta\varphi_1 e^{+kz} e^{i(kx - \omega t)} \quad (2.435)$$

where we dropped the index + for simplicity. Equivalently, we have for fluid 2

$$\varphi_2(x, z, t) = \Delta\varphi_2 e^{-kz} e^{i(kx - \omega t)} \quad (2.436)$$

for which we dropped the positive exponential to avoid a divergence at $z \rightarrow +\infty$. We have therefore three unknown amplitudes to determine our complete solution $\Delta\eta$, $\Delta\varphi_1$ and $\Delta\varphi_2$.

Kinematic condition

We now write the kinematic conditions between each fluid and the interface.

$$\frac{\partial \eta}{\partial t} + v_x \frac{\partial \eta}{\partial x} = v_z \quad (2.437)$$

This equation is evaluated at $z = \eta$ and linearized by Taylor expanding the exponential term and dropping all high-order (quadratic or higher) terms in the equation.

$$\frac{\partial \eta}{\partial t} \simeq \frac{\partial \varphi_1}{\partial z} \quad \text{and} \quad \frac{\partial \eta}{\partial t} \simeq \frac{\partial \varphi_2}{\partial z} \quad (2.438)$$

Using the adopted solutions, we get the following relations between the amplitudes

$$\boxed{-i\omega \Delta\eta = +k \Delta\varphi_1 \quad \text{and} \quad -i\omega \Delta\eta = -k \Delta\varphi_2} \quad (2.439)$$

Bernoulli theorem and pressure equality

We now write the second Bernoulli theorem in each domain as

$$\frac{\partial \varphi_1}{\partial t} + \frac{v_1^2}{2} + \frac{P_1}{\rho_1} + gz = C_1 \quad (2.440)$$

$$\frac{\partial \varphi_2}{\partial t} + \frac{v_2^2}{2} + \frac{P_2}{\rho_2} + gz = C_2 \quad (2.441)$$

The two Bernoulli constants can be found by noticing that the pressure at infinity matches the initial equilibrium model. In other words, the perturbations vanish at infinity. We thus get

$$\frac{P_1}{\rho_1} + gz \rightarrow \frac{P_0}{\rho_1} = C_1 \quad \text{as } z \rightarrow -\infty \quad (2.442)$$

and similarly

$$\frac{P_2}{\rho_2} + gz \rightarrow \frac{P_0}{\rho_2} = C_2 \quad \text{as } z \rightarrow +\infty \quad (2.443)$$

We now linearize the two Bernoulli equations by dropping the kinetic energy term and evaluate them at the interface $z = \eta$. This gives us

$$\frac{\partial \varphi_1}{\partial t} + \frac{P_1}{\rho_1} + g\eta = \frac{P_0}{\rho_1} \quad (2.444)$$

$$\frac{\partial \varphi_2}{\partial t} + \frac{P_2}{\rho_2} + g\eta = \frac{P_0}{\rho_2} \quad (2.445)$$

We finally require $P_1 = P_2$ at the interface, which gives the final relation between the amplitudes

$$\boxed{\rho_1 (-i\omega \Delta \varphi_1 + g \Delta \eta) = \rho_2 (-i\omega \Delta \varphi_2 + g \Delta \eta)} \quad (2.446)$$

Injecting the two previous equations into the third one, we get the dispersion relation as

$$\rho_1 (-\omega^2 + gk) = \rho_2 (+\omega^2 + gk) \quad (2.447)$$

and finally

$$\boxed{\omega^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} gk} \quad (2.448)$$

- If $\rho_1 > \rho_2$, the previous dispersion relation yields stable propagating waves, with velocity $\omega/k \simeq \sqrt{g/k}$. This is the solution for deep sea water, with long wavelength perturbations moving faster than small wavelength ones.
- If $\rho_1 < \rho_2$, on the other hand, the solutions are unstable. This is the *Rayleigh-Taylor instability*. The amplitude of the interface will grow exponentially fast. In the non-linear regime, the low density fluid at the bottom generates bubbles that rise buoyantly in the high density fluid. The high density fluid on top will sink in the form of cold fingers that eventually break into droplets.

Note that the two fluids pressures at the interface are equal and a more general criterion for the instability can be expressed as

$$S_1 = \frac{P_1}{\rho_1^\gamma} > S_2 = \frac{P_2}{\rho_2^\gamma} \quad (2.449)$$

This instability is also called the *interchange instability*, because the high entropy fluid at the bottom will change its position with the low entropy fluid on top. This instability is at the origin of a fundamental process in stars and planets called *convection*. Once the amplitude of the perturbations are large, the instability enters the non-linear regime. Figure 2.9 shows a time sequence of the non-linear development of the Rayleigh-Taylor instability. Initially, the interface is almost perfectly horizontal. Once the amplitude of the instability is large enough, we can see by eye the planar wave solution. Later on, the lighter fluid develops large bubble that rise, while the heavier fluid forms cold and dense fingers that sink in the direction of gravity. One can also see on the sides of the fingers small vortices due to a secondary Kelvin-Helmholtz instability.

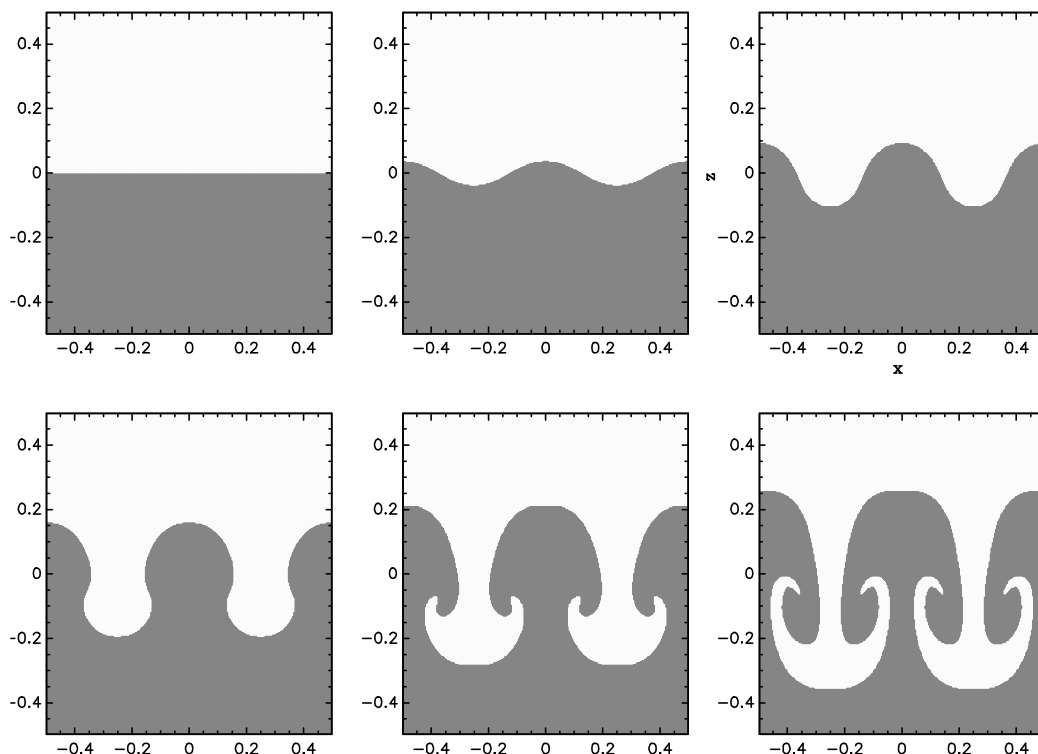


Figure 2.9: Non-linear evolution of the Rayleigh-Taylor instability. Time advances from left to right and from top to bottom. The heavy fluid in white is above the lighter one.

2.10.3 Kelvin-Helmholtz instability

We consider now a slightly different case, with two fluids separated by an horizontal interface, with different densities ρ_1 and ρ_2 and different transverse velocities.

$$\mathbf{v}_1 = V_1 \mathbf{e}_x \quad \text{and} \quad \mathbf{v}_2 = V_2 \mathbf{e}_x \quad (2.450)$$

We ignore gravity so that the initial pressure is uniform in both fluids and equal to P_0 . These initial conditions satisfies trivially the Euler equations and they define our initial equilibrium. We see that in each domain the initial velocity is obviously curl-free. It will remain so after we perturb the interface with a planar wave of the form.

$$\eta(x, t) = \Delta \eta e^{i(kx - \omega t)} \quad (2.451)$$

The velocity perturbations are small enough to be subsonic. Our perturbed velocity fields will be curl-free and divergence-free (incompressible limit). We thus have a potential flow and the scalar potentials satisfy Laplace equation in each domain. We can directly write the solution in the following form

$$\mathbf{v}_1 = V_1 \mathbf{e}_x + \nabla \varphi_1 \quad \text{where} \quad \varphi_1(x, z, t) = \Delta \varphi_1 e^{+kz} e^{i(kx - \omega t)} \quad (2.452)$$

$$\mathbf{v}_2 = V_2 \mathbf{e}_x + \nabla \varphi_2 \quad \text{where} \quad \varphi_2(x, z, t) = \Delta \varphi_2 e^{-kz} e^{i(kx - \omega t)} \quad (2.453)$$

Kinematic conditions

For fluid 1, the kinematic condition writes

$$\frac{\partial \eta}{\partial t} + v_{x,1} \frac{\partial \eta}{\partial x} = v_{z,1} \quad (2.454)$$

with now

$$v_{x,1} = V_1 + \frac{\partial \varphi_1}{\partial x} \quad \text{and} \quad v_{z,1} = \frac{\partial \varphi_1}{\partial z} \quad (2.455)$$

Linearizing this equation at the interface position $z = \eta$ leads us to keep more terms than in the Rayleigh-Taylor case

$$\frac{\partial \eta}{\partial t} + V_1 \frac{\partial \eta}{\partial x} = \frac{\partial \varphi_1}{\partial z} \quad (2.456)$$

We now inject our Ansatz and obtain our first relation between the perturbation amplitudes

$$\boxed{(-i\omega + ikV_1) \Delta \eta = +k \Delta \varphi_1} \quad (2.457)$$

Similarly, for fluid 2, we get

$$\boxed{(-i\omega + ikV_2) \Delta \eta = -k \Delta \varphi_2} \quad (2.458)$$

Bernoulli equation and pressure equilibrium

We now use Bernoulli equation in fluid 1 for which we have no gravity

$$\frac{\partial \varphi_1}{\partial t} + \frac{1}{2} v_{x,1}^2 + \frac{1}{2} v_{z,1}^2 + \frac{P_1}{\rho_1} = C_1 \quad (2.459)$$

We find the Bernoulli constant by requiring that the perturbations vanish at $-\infty$ so that

$$\frac{1}{2} V_1^2 + \frac{P_0}{\rho_1} = C_1 \quad (2.460)$$

Finally, we linearize the Bernoulli equation, dropping all quadratic (or higher order) terms

$$\frac{\partial \varphi_1}{\partial t} + \frac{1}{2} V_1^2 + V_1 \frac{\partial \varphi_1}{\partial x} + \frac{P_1}{\rho_1} = \frac{1}{2} V_1^2 + \frac{P_0}{\rho_1} \quad (2.461)$$

This lead to the pressure in fluid 1 at the interface to be

$$P_1 = P_0 - \rho_1 \left(\frac{\partial \varphi_1}{\partial t} + V_1 \frac{\partial \varphi_1}{\partial x} \right) \quad (2.462)$$

and similarly for fluid 2, we have

$$P_2 = P_0 - \rho_2 \left(\frac{\partial \varphi_2}{\partial t} + V_2 \frac{\partial \varphi_2}{\partial x} \right) \quad (2.463)$$

Imposing pressure equality at the interface and computing the time and space derivatives gives us our third relation

$$\boxed{\rho_1 (-i\omega + ikV_1) \Delta \varphi_1 = \rho_2 (-i\omega + ikV_2) \Delta \varphi_2} \quad (2.464)$$

Combining it with the two previously found relation leads to the dispersion relation

$$\rho_1 (\omega - kV_1)^2 = -\rho_2 (\omega - kV_2)^2 \quad (2.465)$$

This is a problematic equation, because it states that a square is equal to minus another square. The solution is to introduce an imaginary component, which is, as we now know very well, a clear sign of instability.

$$\sqrt{\rho_1} (\omega - kV_1) = \pm i \sqrt{\rho_2} (\omega - kV_2) \quad (2.466)$$

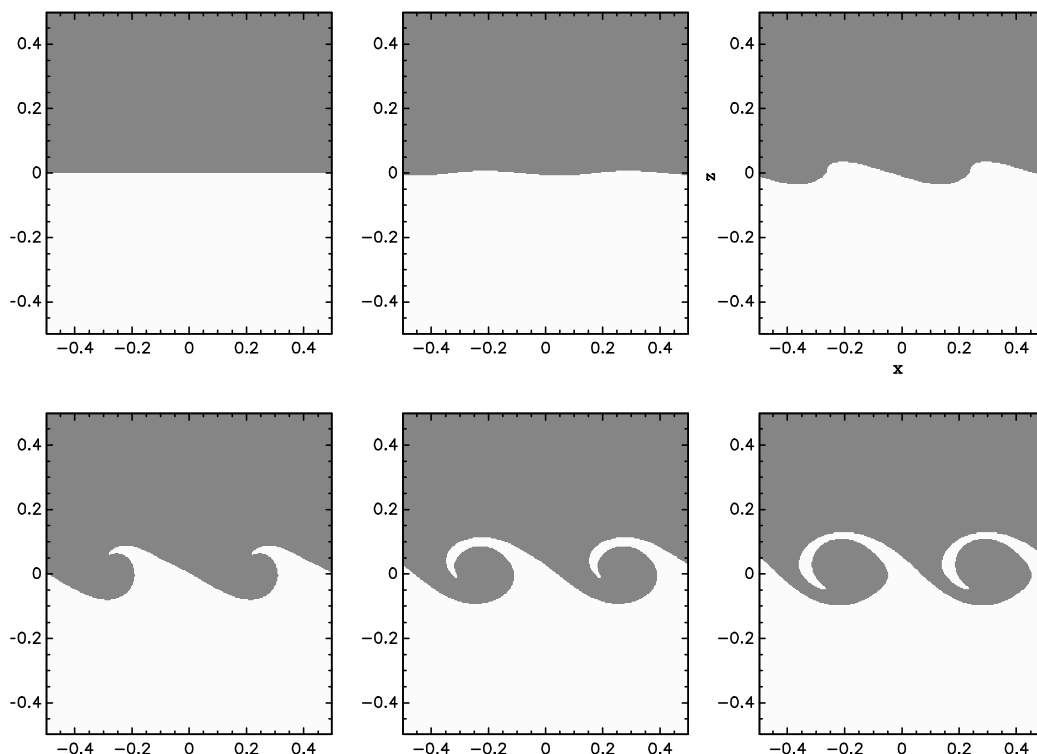


Figure 2.10: Non-linear evolution of the Kelvin-Helmoltz instability. Time advances from left to right and from top to bottom. The heavy fluid is now below the lighter one.

After some manipulations, we obtain the final dispersion relation

$$\boxed{\frac{\omega}{k} = \frac{\rho_1 V_1 + \rho_2 V_2}{\rho_1 + \rho_2} \pm i \frac{\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2} (V_1 - V_2)}$$
 (2.467)

The resulting waves are propagating waves, with speed equal to the density-weighted mean of the two initial velocities, and with amplitude growing exponentially at the rate

$$\gamma \simeq |V_1 - V_2| k$$
 (2.468)

This is the Kelvin-Helmoltz instability. We see that the instability is triggered as soon as the velocity difference is non-zero. This happens all the time! This is also called the *shear instability*. Shearing flow configurations lead systematically to the fast amplification of any small perturbations and to the onset of turbulence. Note that gravity can stabilize the flow if the heavy fluid sits below the lighter one. This is what happens in non self-gravitating accretion disks, for which differential rotation alone does not trigger Kelvin-Helmoltz instability.

During its non-linear evolution (see Figure 2.10), the Kelvin-Helmoltz instability develops complex features such as breaking waves, that later become turbulent eddies. Larger eddies can in turn develop secondary eddies and lead to very complex flow patterns that we call *turbulence*. In conclusion, we have seen two classical equilibrium situations where turbulence forms naturally. We now describe in greater details the mathematical theory of turbulent flows.

2.10.4 Mean-flow equations for turbulent flows

In the last two sections, we have seen how turbulence can emerge from unstable situations. Once the perturbations are fully developed into the non-linear regime, we need a theory to describe

these fluctuations. The mean-flow equations are new fluid equations that can be used to describe the large-scale flow, averaging over the small-scale fluctuations. Density, velocity and pressure fluctuations are treated as random variables, whose exact values are not important. Only the average statistical properties of the fluctuations are important. The new mathematical key concept we introduce in this section is the *averaging operator*. For example, if you consider the density field $\rho(\mathbf{x}, t)$ as the variable of interest, you can define the *space average* within a fixed volume V_0 as

$$\bar{\rho}(t) = \frac{1}{V_0} \int_{V_0} \rho(\mathbf{x}, t) dV \quad (2.469)$$

with the overbar indicating the volume average. You can also use a small probe that records the value of the density at a fixed location, and then you can perform the *time average* within a fixed time interval as

$$\bar{\rho}(\mathbf{x}) = \frac{1}{T_0} \int_{T_0} \rho(\mathbf{x}, t) dt \quad (2.470)$$

These two averaging operators are easy to understand but they contain a fundamental flaw: In both cases, we have lost one independent variable, either time or space, so we won't be able to find new fluid equations. There is one more definition we can use to define the averaging operator, namely the *ensemble average*, defined by

$$\bar{\rho}(\mathbf{x}, t) = \frac{1}{N} \sum_{i=1, N} \rho_i(\mathbf{x}, t) \quad (2.471)$$

Index i now stands for a parallel universe where we have the same exact fluid flow but with different random properties, so we can now average over N parallel universes to get the average and still keep both time and space as independent variables. Although this definition of the averaging operator is practical in a mathematical sense, it is not realistic as one cannot design an experiment over parallel universes we don't know how to explore. We nevertheless assume that these three different definitions are equivalent and lead to the same value for the average quantity. This is called the *ergodic principle*. In what follows, we will use the multiple universes viewpoint, but keep in mind that this can also mean time and/or space average within one single experiment.

Another interpretation of the averaging operator is related to the concept of *smoothing*. We define a smoothing kernel of size R , say a Gaussian filter or a Top-Hat function W , and we define the average at time t and position \mathbf{x} as the convolution of the original density field and the filter function

$$\bar{\rho}(\mathbf{x}, t) = \int_{\mathbb{R}^3} \rho(\mathbf{x}', t) W\left(\frac{\mathbf{x} - \mathbf{x}'}{R}\right) d^3x' \quad (2.472)$$

The average density field is the smoothed density field at scales $r > R$, and the fluctuations are defined as the small-scale perturbations for scales $r < R$. Whatever viewpoint we adopt, we always define the fluctuating density as the different between the average and the original one. We thus have in each parallel universe

$$\rho(\mathbf{x}, t) = \bar{\rho}(\mathbf{x}, t) + \rho'(\mathbf{x}, t) \quad (2.473)$$

$$\mathbf{v}(\mathbf{x}, t) = \bar{\mathbf{v}}(\mathbf{x}, t) + \mathbf{v}'(\mathbf{x}, t) \quad (2.474)$$

$$P(\mathbf{x}, t) = \bar{P}(\mathbf{x}, t) + P'(\mathbf{x}, t) \quad (2.475)$$

If we apply the averaging operator to the previous three equations, we find

$$\bar{\rho}' = 0, \quad \bar{\mathbf{v}}' = 0 \quad \text{and} \quad \bar{P}' = 0 \quad (2.476)$$

Let's now apply this operator to the mass conservation equation. We use in what follows Einstein notations, and start with

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_j) = 0 \quad (2.477)$$

We decompose both the density and the velocity into the mean flow and the fluctuating flow as $\rho = \bar{\rho} + \rho'$ and $v_j = \bar{v}_j + v'_j$ and obtain

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{v}_j + \bar{\rho} v'_j + \rho' \bar{v}_j + \rho' v'_j) = 0 \quad (2.478)$$

We now apply the average operator on top of the entire equation. Already averaged quantity can be considered as constant. Using the multiple universes interpretation, the average operator obviously commutes with time and space derivatives. We therefore get

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{v}_j + \overline{\rho' v'_j}) = 0 \quad (2.479)$$

We see that we have identified a new quantity $\overline{\rho' v'_j} \neq 0$ called *the turbulent mass flux*. Apart from this annoying term, we have derived a new mass conservation equation for the mean flow only, that we can use to solve the mean flow evolution, ignoring the details of the turbulent fluctuations. We however need to come up with a model for the turbulent mass flux.

Another approach, more suited for compressible fluids, is based on the *Favre average*. Favre averaging is based on a mass-weighted average of all specific quantities like the velocity, the specific internal energy or the temperature. We have by definition

$$\bar{\rho} \tilde{\mathbf{v}} = \overline{\rho \mathbf{v}} \quad (2.480)$$

where the tilde means Favre average. We need to define new fluctuations with respect to this new average

$$\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{v}'' \quad \text{and} \quad \epsilon = \tilde{\epsilon} + \epsilon'' \quad (2.481)$$

where now the double prime stands for fluctuations defined with respect to the mass-weighted, Favre average, and single prime for fluctuations with respect to the standard, volume-weighted average. Now we have for the fluctuating part the following property

$$\overline{\rho \mathbf{v}''} = 0 \quad \text{and} \quad \overline{\rho \epsilon''} = 0 \quad (2.482)$$

but of course we have unfortunately in this case

$$\overline{\mathbf{v}''} \neq 0 \quad \text{and} \quad \overline{\epsilon''} \neq 0 \quad (2.483)$$

Let's re-derive the mass conservation equation using the Favre average. This time, it is important not to develop the density inside the space derivative

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_j} (\rho \tilde{v}_j + \rho v''_j) = 0 \quad (2.484)$$

We now apply the average operator to the entire equation, which leads now to the much simpler form

$$\boxed{\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\rho} \tilde{v}_j) = 0} \quad (2.485)$$

By adopting a new frame of reference, we have absorbed the turbulent mass flux in the Favre average. We can now relate the two different definitions of the mean flow velocity by simply identifying terms

$$\tilde{v}_j = \bar{v}_j + \frac{\overline{\rho'v_j'}}{\bar{\rho}} \quad (2.486)$$

We now move to the momentum conservation equation. We start as always with

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (\rho v_i v_j) + \frac{\partial P}{\partial x_i} = \rho g_i \quad (2.487)$$

We decompose the velocity into $v_j = \tilde{v}_j + v_j''$ but not the density

$$\frac{\partial}{\partial t} (\rho \tilde{v}_i) + \frac{\partial}{\partial t} (\rho v_i'') + \frac{\partial}{\partial x_j} (\rho \tilde{v}_i \tilde{v}_j + \tilde{v}_i \rho v_j'' + \rho v_i'' \tilde{v}_j + \rho v_i'' v_j'') + \frac{\partial P}{\partial x_i} = \rho g_i \quad (2.488)$$

Now we apply the average operator over the entire equation. All terms with a $\overline{\rho v_i''} = 0$ vanish and we get the momentum conservation equation of the mean flow

$$\boxed{\frac{\partial}{\partial t} (\bar{\rho} \tilde{v}_i) + \frac{\partial}{\partial x_j} (\bar{\rho} \tilde{v}_i \tilde{v}_j + \overline{\rho v_i'' v_j''}) + \frac{\partial \bar{P}}{\partial x_i} = \bar{\rho} g_i} \quad (2.489)$$

This equation looks very much like the original one, but for one extra term, called *the turbulent pressure tensor* defined as

$$\mathbb{P}_T = \overline{\rho \mathbf{v}'' \otimes \mathbf{v}''} \quad (2.490)$$

In analogy with the chapter on kinetic theory, we decompose this tensor into a traceless tensor and an isotropic tensor as

$$\mathbb{P}_T = \mathbb{R}_T + P_T \mathbb{I} \quad (2.491)$$

where the turbulent pressure is defined by

$$\boxed{P_T = \frac{1}{3} \overline{\rho v''^2} = \frac{2}{3} K_T} \quad (2.492)$$

and we have introduced the turbulent kinetic energy K_T . The Reynolds tensor is now defined by

$$\boxed{\mathbb{R}_T = \overline{\rho \mathbf{v}'' \otimes \mathbf{v}''} - P_T \mathbb{I}} \quad (2.493)$$

and the momentum conservation equation for the mean flow can be modified as follows

$$\frac{\partial}{\partial t} (\bar{\rho} \tilde{v}_i) + \frac{\partial}{\partial x_j} (\bar{\rho} \tilde{v}_i \tilde{v}_j) + \frac{\partial}{\partial x_j} (R_{T,ij}) + \frac{\partial \bar{P}}{\partial x_i} + \frac{\partial P_T}{\partial x_i} = \bar{\rho} g_i \quad (2.494)$$

Finally, we derive the total energy equation for the mean flow. We start with the original energy equation in Eulerian form

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} (E + P) v_j = \rho v_j g_j \quad (2.495)$$

The total energy is decomposed into

$$E = \frac{1}{2} \rho v^2 + \rho \epsilon = \frac{1}{2} \rho \tilde{v}^2 + \rho \tilde{v}_i v_i'' + \frac{1}{2} \rho v''^2 + \rho \tilde{\epsilon} + \rho \epsilon'' \quad (2.496)$$

so that averaging the time derivative leads to

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \bar{\rho} \tilde{v}^2 + 0 + \frac{1}{2} \overline{\rho v''^2} + \bar{\rho} \tilde{\epsilon} + 0 \right) = \frac{\partial E_{\text{tot}}}{\partial t} \quad (2.497)$$

where now the total energy writes

$$\boxed{E_{\text{tot}} = \frac{1}{2}\bar{\rho}\tilde{v}^2 + \bar{\rho}\tilde{\epsilon} + K_T} \quad (2.498)$$

We see that this new total energy is equal to the kinetic energy of the mean flow, plus the internal energy of the mean flow, plus the turbulent kinetic energy. Let's now compute the space derivatives. Using $P = (\gamma - 1)\rho\epsilon$, we have

$$\frac{\partial}{\partial x_j} (E + P) v_j = \frac{\partial}{\partial x_j} \left(\frac{1}{2}\bar{\rho}\tilde{v}^2 v_j + \bar{\rho}\tilde{v}_i v_i'' v_j + \frac{1}{2}\bar{\rho}v''^2 v_j + \gamma\bar{\rho}\tilde{\epsilon} v_j + \gamma\bar{\rho}\epsilon'' v_j \right) \quad (2.499)$$

We now decompose $v_j = \tilde{v}_j + v_j''$ in the previous equation and apply immediately the average operator to obtain

$$\frac{\partial}{\partial x_j} \left(\frac{1}{2}\bar{\rho}\tilde{v}^2 \tilde{v}_j + 0 + 0 + \overline{\rho v_i'' v_j''} \tilde{v}_i + \frac{1}{2}\overline{\rho v''^2} \tilde{v}_j + \frac{1}{2}\overline{\rho v''^2 v_j''} + \gamma\bar{\rho}\tilde{\epsilon} \tilde{v}_j + 0 + 0 + \overline{\gamma\rho\epsilon'' v_j''} \right) \quad (2.500)$$

We recognize the total energy flux due to the mean flow, the turbulence pressure tensor, that we decompose into the Reynolds stress tensor plus the turbulent pressure, and a new term called *the turbulent heat flux*. The total energy equation for the mean flow finally writes

$$\boxed{\frac{\partial E_{\text{tot}}}{\partial t} + \frac{\partial}{\partial x_j} (E_{\text{tot}} + P_{\text{tot}}) \tilde{v}_j + \frac{\partial}{\partial x_j} (R_{T,ij} \tilde{v}_i) + \frac{\partial}{\partial x_j} (Q_{T,j}) = \bar{\rho} \tilde{v}_j g_j} \quad (2.501)$$

where the total pressure is defined by

$$\boxed{P_{\text{tot}} = \bar{P} + P_T} \quad (2.502)$$

and the turbulent heat flux by

$$\boxed{\mathbf{Q}_T = \frac{1}{2}\overline{\rho v''^2 \mathbf{v}''} + \gamma\overline{\rho\epsilon'' \mathbf{v}''}} \quad (2.503)$$

The analogy with kinetic theory is striking. We see that we have now introduced a new scale which captures *mesoscopic processes* such as turbulence, between the macroscopic scale of the system as a whole, described by the mean flow equations, and the microscopic processes such as collisions. Turbulence is captured by the averaging operator in pretty much the same way particles are captured in phase space by integrating over velocity space. We are facing the same problem than in kinetic theory: We don't know the exact form of the Reynolds stress tensor and the turbulent heat flux. We don't know the value of the turbulent kinetic energy either. We will try to remedy for the latter in the next section.

2.10.5 Turbulent kinetic energy equation

We want to derive from first principles an equation governing the time evolution of the turbulent kinetic energy K_T . For this, we start as always from the Euler equations,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_j) = 0 \quad (2.504)$$

$$\rho \frac{\partial}{\partial t} v_i + \rho v_j \frac{\partial}{\partial x_j} v_i + \frac{\partial P}{\partial x_i} = \rho g_i \quad (2.505)$$

In the latter equation, we decompose $v_i = \tilde{v}_i + v_i''$ but (very important!) not v_j . We obtain

$$\rho \frac{\partial}{\partial t} \tilde{v}_i + \rho \frac{\partial}{\partial t} v_i'' + \rho v_j \frac{\partial}{\partial x_j} \tilde{v}_i + \rho v_j \frac{\partial}{\partial x_j} v_i'' + \frac{\partial P}{\partial x_i} = \rho g_i \quad (2.506)$$

We now multiply the previous equation by v_i'' and sum over the three directions. If possible, we make the specific kinetic energy appear. Otherwise, Einstein's summation rules apply.

$$\rho v_i'' \frac{\partial}{\partial t} \tilde{v}_i + \rho \frac{\partial}{\partial t} \left(\frac{1}{2} v''^2 \right) + \rho v_i'' v_j \frac{\partial}{\partial x_j} \tilde{v}_i + \rho v_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} v''^2 \right) = -v_i'' \frac{\partial P}{\partial x_i} + \rho v_i'' g_i \quad (2.507)$$

We can now add to the previous equation

$$+ \frac{1}{2} v''^2 \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_j) \right] \quad (2.508)$$

since it is exactly zero owing to the mass conservation equation. We get the slightly simpler form

$$\rho v_i'' \frac{\partial}{\partial t} \tilde{v}_i + \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v''^2 \right) + \rho v_i'' v_j \frac{\partial}{\partial x_j} \tilde{v}_i + \frac{\partial}{\partial x_j} \left(\frac{1}{2} \rho v''^2 v_j \right) = -v_i'' \frac{\partial P}{\partial x_i} + \rho v_i'' g_i \quad (2.509)$$

Now comes the fateful moment: We at last decompose v_j into $\tilde{v}_j + v_j''$ and apply immediately the averaging operator to the entire equation.

$$0 + \frac{\partial}{\partial t} K_T + 0 + \overline{\rho v_i'' v_j''} \frac{\partial}{\partial x_j} \tilde{v}_i + \frac{\partial}{\partial x_j} (K_T \tilde{v}_j) + \frac{\partial}{\partial x_j} \left(\frac{1}{2} \overline{\rho v''^2 v_j''} \right) = -v_i'' \overline{\frac{\partial P}{\partial x_i}} + 0 \quad (2.510)$$

We recognize again the turbulent pressure tensor, that we decompose again into the isotropic turbulent pressure plus the Reynolds stress tensor. We also recognize one component of the heat flux, the one associated to the turbulent transport of turbulent kinetic energy. We note it here \mathbf{Q}'_T . Finally, on the right-hand side, we have a new term called the *convective term* or the *turbulent work of the pressure fluctuations*. In summary, we have for the turbulent kinetic energy equation

$$\frac{\partial}{\partial t} K_T + \frac{\partial}{\partial x_j} (K_T \tilde{v}_j) + P_T \frac{\partial \tilde{v}_j}{\partial x_j} = -R_{T,ij} \frac{\partial \tilde{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j} (Q'_{T,j}) - v_j'' \overline{\frac{\partial P}{\partial x_j}} \quad (2.511)$$

Here again, this equation looks very familiar. On the left-hand side, we have the transport of turbulent kinetic energy by the mean flow, as well as the $p dV$ work of the turbulent pressure due to the mean flow. This is almost identical to the internal energy equation for the fluid. On the right-hand side, we have new terms that are associated to the production of turbulent kinetic energy. They are actually called the *production terms*. We will see in the next section how one can compute these terms within some clever approximations. Qualitatively, however, these terms will inject turbulent kinetic energy in the system, but we have no way to dissipate this energy. In fact, viscous processes, not discussed here, will transform turbulent kinetic energy into heat (see the vorticity equation in one of the previous sections). To properly close the system, we need to add a new term called the *dissipation term*, which encodes the viscous processes at small scales and acts as a sink term in the turbulent kinetic energy equation

$$\boxed{\frac{\partial}{\partial t} K_T + \frac{\partial}{\partial x_j} (K_T \tilde{v}_j) + P_T \frac{\partial \tilde{v}_j}{\partial x_j} = -R_{T,ij} \frac{\partial \tilde{v}_i}{\partial x_j} - \frac{\partial}{\partial x_j} (Q'_{T,j}) - v_j'' \overline{\frac{\partial P}{\partial x_j}} - \epsilon_T} \quad (2.512)$$

This new term ϵ_T can also be computed using the averaging operator, if one includes viscosity in the Euler equations since the beginning of the present derivation. We leave the description of turbulence dissipation to the last section of the chapter. This equation, sometimes called the K-equation, can be augmented by a similar equation for the dissipation term ϵ_T . The resulting model is very popular in engineering, and is called the K- ϵ model. In astrophysics, the most popular approach is probably the *mixing length theory*.

2.10.6 Eddy-viscosity and mixing length theory

We discuss now a model that gives an explicit form for the Reynolds tensor introduced above. This model is called the *eddy-viscosity model* or the *Boussinesq approximation*. The idea is to push the analogy with kinetic theory one step further and identify turbulent eddies with microscopic particles. In kinetic theory, particles move with random motions, at a typical speed given by the sound speed a and over a typical length scale given by the collision mean free path λ_{coll} . In the eddy-viscosity model, eddies also move with random motions, at a typical speed given by the turbulent velocity dispersion σ_T and over a typical length scale called *the mixing length* and noted ℓ .

The difference between particles and eddies is mostly the nature of the dissipation process. For turbulence, kinetic energy dissipation occur through secondary instabilities, similar to what we have seen in the Rayleigh-Taylor instability, that trigger a *turbulent cascade* to smaller and smaller scales until the typical eddy size becomes comparable to the collision mean free path. The kinetic energy is then dissipated into heat through viscous processes. The analogy then leads logically to adopt for the Reynold tensor the same form that the one we obtained using Chapman-Enskog theory, namely

$$\mathbb{R}_T = -\mu_T \tilde{\mathbb{S}} \quad (2.513)$$

where the shear tensor is now defined for the mean flow as

$$\tilde{S}_{ij} = \frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_i} - \frac{2}{3} (\nabla \cdot \tilde{\mathbf{v}}) \delta_{ij} \quad (2.514)$$

The turbulent viscosity coefficient is given following the same analogy by

$$\mu_T = \bar{\rho} \ell \sigma_T \quad (2.515)$$

We can then compute the production term associated to the Reynolds tensor using the dissipation function of the mean flow (see chapter 1) which writes

$$\tilde{\Phi} = \mu_T \left(\tilde{\mathbb{S}} : \nabla \tilde{\mathbf{v}} \right) = 2\mu_T \sum_{ij} \left[\frac{1}{2} \left(\frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_i} \right) - \frac{1}{3} (\nabla \cdot \tilde{\mathbf{v}}) \delta_{ij} \right]^2 \quad (2.516)$$

It is customary to write the dissipation function using Einstein's summation rule as

$$\tilde{\Phi} = \frac{1}{2} \mu_T \left| \tilde{S}_{ij} \right|^2 \quad (2.517)$$

We now consider a mean equilibrium flow dominated by strong shear and in absence of gravity. In this case, we can ignore the mean pressure gradients, as it is usually a condition for equilibrium. More importantly, we can also assume that the pressure fluctuations are also zero. If a small eddy or a low density bubble in the fluid is over- or under-pressurised compared to the mean flow pressure, it will very quickly expand or collapse until it reaches pressure equilibrium. Another viewpoint is to appeal to the Rankine-Hugoniot relations for a contact discontinuity which demand pressure equilibrium at the interface between the eddy or the bubble and the ambient gas. Given these conditions, we can safely assume that the convective production term in the turbulent kinetic energy equation is zero. We also neglect the turbulent kinetic energy heat flux $\mathbf{Q}'_T \simeq 0$ as it is of higher order (third-order) in the velocity fluctuations than the shear production term. As a consequence, the turbulent kinetic energy equation can be written as

$$\frac{\partial}{\partial t} K_T + \frac{\partial}{\partial x_j} (K_T \tilde{v}_j) + P_T \frac{\partial \tilde{v}_j}{\partial x_j} = \frac{1}{2} \mu_T \left| \tilde{S}_{ij} \right|^2 - \epsilon_T \quad (2.518)$$

In the *mixing length theory*, we consider that eddies will travel over the mixing length at a typical velocity σ_T before releasing their kinetic energy to viscous processes and heat. We can thus write a simple model for ϵ_T as

$$\epsilon_T = \frac{K_T}{\tau_{\text{diss}}} \quad \text{with} \quad \tau_{\text{diss}} = \frac{\ell}{\sigma_T} \quad \text{and} \quad K_T = \frac{1}{2} \bar{\rho} \sigma_T^2 \quad (2.519)$$

The mixing length theory then considers that there is a balance between production and dissipation of turbulence. Both terms exactly cancel out in the right-hand side of the turbulent kinetic energy equation. This is also called the *Smagorinsky approximation*. Using $\mu_T = \bar{\rho} \ell \sigma_T$, we get

$$\frac{1}{2} \bar{\rho} \ell \sigma_T \left| \tilde{S}_{ij} \right|^2 = \frac{K_T}{\ell} \sigma_T \quad \text{which leads to} \quad \boxed{\sigma_T = \ell \sqrt{\left| \tilde{S}_{ij} \right|^2}} \quad (2.520)$$

In conclusion, the turbulence velocity dispersion is fully specified as the result of the balance between injection of turbulence due to the mean shear flow (also known as the Kelvin-Helmoltz instability) and dissipation of turbulence due to mixing and viscous processes. The only remaining free parameter is the mixing length, that can be adjusted to an experiment, or identified with the smoothing scale of the corresponding averaging operator. In the latter case, the theory is called *subgrid-scale* (SGS) turbulence.

Note that in presence of gravity, the present approach is not valid, as gravity will stabilise the Kelvin-Helmoltz instability and prevent the production of turbulence, unless shear stress is strong enough to overcome gravity. In Keplerian disks, for example, the shear stress is the product of the differential rotation, so we know that gravity will always stabilise the flow. The source of turbulence has to be found elsewhere. Around supermassive black holes, the source of turbulence is due to magnetic fields, In this case, although the eddy-viscosity model is a good approximation, one cannot apply the Smagorinsky model to determine σ_T , since the production term will be completely different. For protoplanetary disks, where magnetic fields are weak or absent, the production term is unknown and could well be zero.

2.10.7 Convective heat flux in stars and planets

We now consider a different type of equilibrium for the mean flow, the hydrostatic equilibrium, for which we have in spherical systems like stars or planets

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{P}}{\partial r} = -g \quad \text{and} \quad \tilde{\mathbf{v}} = 0 \quad (2.521)$$

The gravity is purely radial and pointing downward. We assume that the equilibrium structure of the star follows a polytropic relation, as the result of an equilibrium between various heating and cooling processes

$$\bar{P} = A \bar{\rho}^\Gamma \quad (2.522)$$

In the same time, the gas follows the ideal gas equation of state with

$$P = (\gamma - 1)e \quad (2.523)$$

We now want to perturb slightly this perfect equilibrium, creating velocity, density and temperature fluctuations. Following the same argument used in the previous section, pressure equilibrium will be maintained within the fluctuations so that $P' = 0$, owing to the contact discontinuity jump relations. In this context dominated by gravity, fluctuations are closer to the Rayleigh-Taylor picture and are not really eddies but underdense rising bubbles evolving

together with overdense sinking fingers. Since in this hydrostatic case the mean flow velocity is zero, we can write the fluctuating velocity equation as

$$\rho \frac{\partial}{\partial t} v_i'' + \rho v_j \frac{\partial}{\partial r} v_i'' = \rho \frac{D}{Dt} v_i'' = -\frac{\partial P}{\partial r} - \rho g \quad (2.524)$$

The right-hand side can be decomposed into

$$\rho \frac{D}{Dt} v_i'' = -\frac{\partial \bar{P}}{\partial r} - \bar{\rho} g - \rho' g \quad (2.525)$$

where we use the fact that $P' = 0$. Since the mean flow is in hydrostatic equilibrium, we obtain finally

$$\rho \frac{D}{Dt} v_i'' = -\rho' g \quad (2.526)$$

We are left with only one force, namely the *buoyancy* of the fluid.

Convective instability

If we slightly perturb the fluid at some initial radius r_i with $\rho \simeq \bar{\rho}(r_i)$ but $\rho' < 0$, then the bubble will rise owing to the buoyant force. Let's assume that the bubble travels until some final radius r_f . The evolution of the bubble is considered to be fast enough so that the thermal balance between heating and cooling has not enough time to apply. The evolution of the bubble will then be adiabatic and the entropy of the bubble will be conserved

$$\frac{P_f}{\rho_f^\gamma} = \frac{P_i}{\rho_i^\gamma} \quad (2.527)$$

Because of the pressure equilibrium condions, the final bubble density will be

$$\rho_f = \bar{\rho}(r_i) \left(\frac{\bar{P}(r_f)}{\bar{P}(r_i)} \right)^{\frac{1}{\gamma}} \quad (2.528)$$

Writing $\Delta r = r_f - r_i \ll r_i$, we can Taylor expand the previous equation and get

$$\rho_f = \bar{\rho}(r_i) \left(1 + \frac{1}{\gamma} \frac{1}{\bar{P}} \frac{\partial \bar{P}}{\partial r} \Delta r \right) = \bar{\rho}(r_i) \left(1 - \frac{1}{\gamma} \frac{\bar{\rho} g}{\bar{P}} \Delta r \right) \quad (2.529)$$

We used the hydrostatic equation to replace the pressure gradient in the rightmost equation. The equilibrium density, on the other hand, follows the polytropic relation, so we have

$$\bar{\rho}(r_f) = \bar{\rho}(r_i) \left(1 - \frac{1}{\Gamma} \frac{\bar{\rho} g}{\bar{P}} \Delta r \right) \quad (2.530)$$

The bubble will keep on rising due to buoyancy if $\rho_f < \bar{\rho}(r_f)$ or equivalently $\Gamma > \gamma$. This is the criterion for the convective instability. The bubble will not travel and rise continuously towards the outskirts of the star. Following mixing length theory, we consider that after a length $\Delta r = \ell$, the bubble will mix with its surroundings and dissipates its energy. We can finally compute the typical density fluctuations due to rising and mixing bubbles as

$$\rho' = \rho_f - \bar{\rho}(r_f) = \bar{\rho} \left(\frac{1}{\Gamma} - \frac{1}{\gamma} \right) \frac{\bar{\rho} g \ell}{\bar{P}} \quad (2.531)$$

We see that we have indeed $\rho' < 0$ for $\Gamma > \gamma$.

Convective heat flux

When we derived the total energy equation for the mean flow, we have obtained new terms featuring the Reynolds tensor and the turbulent heat flux. In the context of an hydrostatic mean flow, the shear tensor is zero everywhere and we can ignore the Reynold stress. The turbulent heat flux, on the other hand, is very important. In our spherically symmetric star, we have a component only in the radial direction

$$Q_T = \frac{1}{2} \overline{\rho v''^2 v''} + \gamma \overline{\rho \epsilon'' v''} \quad (2.532)$$

In stellar interiors, the velocity fluctuations are highly subsonic with $v'' \ll c_s$. The specific internal energy is by far the dominant energy source. When our rising bubbles reach their terminal radius, set by the mixing length, they deliver to their surroundings mostly internal energy, so that

$$Q_T \simeq \frac{\gamma k_B}{(\gamma - 1)m} \overline{\rho T'' v''} = C_P \overline{\rho T'' v''} \quad (2.533)$$

where C_P is the well-known specific heat capacity at constant pressure. Computing the convective heat flux from first principle is not easy. In what follows, we assume that bubbles are rising until they reach exactly one mixing length, and that the temperature difference between the bubble and the ambient gas at that final radius will set the typical temperature fluctuation

$$T'' = T_f - \tilde{T}(r_f) \quad (2.534)$$

Since our evolution is adiabatic, we can use exactly the same trick as before, Taylor expanding the mean pressure and get

$$T_f = \tilde{T}(r_i) \left(1 - \frac{\gamma - 1}{\gamma} \frac{\bar{\rho} g}{\bar{P}} \Delta r \right) \quad (2.535)$$

while the mean flow temperature follows the polytropic relation

$$\tilde{T}(r_f) = \tilde{T}(r_i) \left(1 - \frac{\Gamma - 1}{\Gamma} \frac{\bar{\rho} g}{\bar{P}} \Delta r \right) \quad (2.536)$$

From this, we get for the typical temperature fluctuation

$$T'' = \tilde{T} \left(\frac{1}{\gamma} - \frac{1}{\Gamma} \right) \frac{\bar{\rho} g \ell}{\bar{P}} \quad (2.537)$$

Note that $T'' > 0$ if $\Gamma > \gamma$. Otherwise, there is no instability and no temperature fluctuations. For the typical velocity fluctuations, we use as before $v'' \simeq \sigma_T$ the turbulence velocity dispersion. We finally get the convective heat flux

$$Q_T \simeq \bar{\rho} C_P \tilde{T} \sigma_T \left(\frac{1}{\gamma} - \frac{1}{\Gamma} \right) \frac{\bar{\rho} g \ell}{\bar{P}} \quad \text{if } \Gamma > \gamma, \text{ otherwise } Q_T = 0. \quad (2.538)$$

Mixing length theory for convection

In order to determine the turbulent velocity dispersion, we need to solve the turbulent kinetic energy equation. The production term associated to the Reynolds tensor is again zero because of our hydrostatic mean flow. What is important here is the convective production term. We thus have

$$\frac{\partial K_T}{\partial t} = -v'' \frac{\partial P}{\partial r} - \epsilon_T \quad (2.539)$$

Exploiting the hydrostatic mean flow equation and the fact that $P' = 0$, we have

$$\frac{\partial P}{\partial r} = \frac{\partial \bar{P}}{\partial r} = -\bar{\rho}g \quad (2.540)$$

and also following the property of the velocity fluctuations v''

$$\rho v'' = \bar{\rho}v'' + \rho'v'' \quad \text{so that} \quad \overline{\rho v''} = 0 = \overline{\bar{\rho}v''} + \overline{\rho'v''} \quad (2.541)$$

we can finally write the turbulent kinetic energy equation as

$$\frac{\partial K_{\text{T}}}{\partial t} = -\overline{\rho'v''g} - \epsilon_{\text{T}} \quad (2.542)$$

Using the previously computed density fluctuations ρ' and velocity fluctuations v'' , we compute the source term for turbulence as

$$-\overline{\rho'v''g} = \bar{\rho}\sigma_{\text{T}}g \left(\frac{1}{\gamma} - \frac{1}{\Gamma} \right) \frac{\bar{\rho}g\ell}{\bar{P}} = \epsilon_{\text{T}} \quad (2.543)$$

We use here again Smagorinsky approximation, imposing a strict balance between turbulence production (through the Rayleigh-Taylor instability or buoyancy) and destruction (through viscosity and heat conduction) over one mixing length. The dissipation term can be written as before,

$$\epsilon_{\text{T}} = \frac{K_{\text{T}}}{\ell} \sigma_{\text{T}} \quad (2.544)$$

which leads to the Smagorinsky prediction for the velocity dispersion of the convective flow

$$\sigma_{\text{T}} = \frac{\bar{\rho}g\ell}{\bar{P}} \sqrt{\frac{2\bar{P}}{\bar{\rho}} \left(\frac{1}{\gamma} - \frac{1}{\Gamma} \right)} \quad \text{if } \Gamma > \gamma, \text{ otherwise } \sigma_{\text{T}} = 0. \quad (2.545)$$

This completely specifies the convective heat flux. Inside stars and planets, heating and cooling processes can be very complex. For example, the core of the Sun is the realm of nuclear reactions, while the radiative zone, as the name indicates, transports the released energy outside through radiation transfer mechanisms. If for some reason, the resulting *local* polytropic index $\Gamma(r)$ always remains smaller than γ , the star will be convectively stable. If, on the other hand, there exists a region $r_1 < r < r_2$ for which $\Gamma(r) > \gamma$, a strong convective flux will develop and redistributes the energy so that $\Gamma(r) \simeq \gamma$ within that region. These regions are called convective zones, and the main property of convective zones is a quasi-uniform entropy. Cooling and heating processes constantly fight against convection to have $\Gamma > \gamma$, but convection fights back to restore $\Gamma = \gamma$. In the Sun, the convective zone occupies roughly half of the volume and almost reaches the surface of the photosphere. In extreme cases, more massive stars can be entirely convective and turbulent.

2.10.8 Kolmogorov theory and Burgers turbulence

After describing in great details the dynamics of the mean flow, we now focus on the properties of the fluctuations themselves. We will present the probability distribution function (PDF) of the velocity and also of the density fluctuations, and more importantly derive the shape of the *energy spectrum* of turbulence. The theory of Kolmogorov will allow us to derive the scale-dependant energy distribution of subsonic turbulence. In the interstellar medium, however, the turbulence is mostly supersonic and is better described using the theory of Burgers.

Gaussian and lognormal distributions

In what follows, we will use the isothermal Euler equations as our guiding model. We also use a frame of reference for which the mean flow velocity is zero. We will therefore drop the prime and double prime notations for the fluctuations. In Lagrangian form, the Euler equations writes

$$\frac{1}{\rho} \frac{D\rho}{Dt} = \frac{D \ln \rho}{Dt} = -\nabla \cdot \mathbf{v} \quad (2.546)$$

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla P = -a^2 \nabla \ln \rho \quad (2.547)$$

Velocity and density fluctuations are considered as random processes. We can model the effect of these fluctuations on the flow by writing a sequence of time step of fixed size Δt , so that

$$\ln \rho^{n+1} = \ln \rho^n - \nabla \cdot \mathbf{v}^n \Delta t \quad (2.548)$$

$$\mathbf{v}^{n+1} = \mathbf{v}^n - a^2 \nabla \ln \rho^n \Delta t \quad (2.549)$$

Summing up since the initial time, we get finally

$$\ln \rho^{n+1} = \ln \rho^0 - \Delta t \sum_{i=1, n} \nabla \cdot \mathbf{v}^i \quad (2.550)$$

$$\mathbf{v}^{n+1} = \mathbf{v}^0 - a^2 \Delta t \sum_{i=1, n} \nabla \ln \rho^i \quad (2.551)$$

Both \mathbf{v} and $\ln \rho$ can be seen as a large sum of independant random variables. We can apply the *central limit theorem* and conclude that both \mathbf{v} and $\ln \rho$ are distributed according to a Gaussian PDF. We define $f(v_x, v_y, v_z)$ the PDF of the velocity fluctuations, so that we can write the probability to have a velocity fluctuation between v_x and $v_x + dv_x$, v_y and $v_y + dv_y$, v_z and $v_z + dv_z$ as

$$dP = f(v_x, v_y, v_z) dv_x dv_y dv_z \quad \text{and} \quad \int_{\mathbb{R}^3} f(\mathbf{v}) d^3v = 1 \quad (2.552)$$

Similarly, we define

$$s = \ln \rho / \bar{\rho} \quad (2.553)$$

and $f(s)$ so that we can write the probability to have a density fluctuation between s and $s + ds$ as

$$dP = f(s) ds \quad \text{and} \quad \int_{\mathbb{R}} f(s) ds = 1 \quad (2.554)$$

The central limit theorem gave us the exact form of these distribution functions, with

$$f(\mathbf{v}) = \frac{1}{(2\pi\sigma_v^2)^{3/2}} \exp\left(-\frac{(\mathbf{v} - \bar{\mathbf{v}})^2}{2\sigma_v^2}\right) \quad (2.555)$$

$$f(s) = \frac{1}{(2\pi\sigma_s^2)^{1/2}} \exp\left(-\frac{(s - \bar{s})^2}{2\sigma_s^2}\right) \quad (2.556)$$

Recall that if we are in the comoving frame of the mean flow, we have $\bar{\mathbf{v}} = 0$. Note that variance σ_v , although related to the mean flow turbulent kinetic energy, is not the same quantity, because σ_T is a mass-weighted quantity, while σ_v is not. Only in the incompressible case do we have the relation

$$\sigma_T^2 = 3\sigma_v^2 \quad \text{if} \quad \rho \equiv \bar{\rho} \quad (2.557)$$

Nevertheless, the analogy with kinetic theory is strengthened, as the Gaussian PDF for the velocity fluctuations is *identical* to the Maxwell-Boltzmann DF for particles in local thermodynamical equilibrium. The density PDF is called a *lognormal* distribution because the log follows a Gaussian distribution. The mean value \bar{s} can be found by requiring

$$\bar{\rho} = \int_{\mathbb{R}} \rho f(s) ds \quad \text{or} \quad \int_{\mathbb{R}} \exp^s f(s) ds = 1 \quad \rightarrow \quad \boxed{\bar{s} = -\frac{\sigma_s^2}{2}} \quad (2.558)$$

We need now to estimate the variance of the lognormal distribution σ_s . For this, we consider that the velocity field can be decomposed into a collection of planar sine waves. In other words, we perform the Fourier transform of the velocity field. We know from a previous section that these planar waves will evolve into shocks in a finite amount of time. These isothermal shocks will compress the density of the fluid, with a compression ratio scaling as the square of the Mach number. If we use the frame for which the mean flow velocity $\bar{\mathbf{v}} = 0$, we can use the compression ratio of the “isothermal shock on a wall” example from the previous sections, with

$$r = 1 + \frac{1}{4} \frac{v_x^2}{a^2} \quad (2.559)$$

We define the Mach number of the turbulent flow using the 1D velocity dispersion

$$\boxed{\mathcal{M} = \frac{\sigma_v}{a}} \quad (2.560)$$

Although the typical average shock compression ratio and the corresponding typical average post-shock density can be computed as

$$r = 1 + \frac{1}{4} \mathcal{M}^2 \quad \text{and} \quad \rho = \bar{\rho} \left(1 + \frac{1}{4} \mathcal{M}^2 \right) \quad (2.561)$$

shock front are usually followed by an exponential rarefaction wave, so that the variance of the lognormal distribution is reduced to

$$\boxed{\sigma_s \simeq \sqrt{\ln \left(1 + \frac{1}{4} \mathcal{M}^2 \right)}} \quad (2.562)$$

Both velocity and density PDFs are therefore fully specified using only one parameter, the velocity dispersion σ_v . We see also that for a flow with a low Mach number, with $\mathcal{M} \rightarrow 0$, the density PDF converges towards a delta function at $\rho = \bar{\rho}$, which is consistent with the fact that the flow is incompressible. For very large Mach number, however, the lognormal PDF results in a very wide range of densities.

Subsonic turbulence and the energy spectrum

Subsonic turbulence is traditionally described using the theory of Kolmogorov. This theory considers that the kinetic energy of turbulent eddies is injected on large scales, say the size of the system of interest, or the mixing length. These turbulent eddies then develop secondary eddies, because of the Kelvin-Helmoltz instability for example, transferring the kinetic energy to smaller and smaller scales. The process repeats itself with smaller and smaller eddies, building up a *turbulent cascade*. Once the smallest eddies reach the size of the collision mean free path, viscous processes finally dissipate the kinetic energy into heat.

In order to turn this qualitative vision into a quantitative theory, we need to introduce the concept of Gaussian random fields. We start with our velocity perturbations, that follows a Gaussian statistics. We define the Fourier transform of the velocity field by

$$\widehat{\mathbf{v}}(\mathbf{k}) = \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{x}) \exp^{i\mathbf{k}\cdot\mathbf{x}} d^3x \quad (2.563)$$

Gaussian random fields follows strict properties in the Fourier domain. If we write the complex number $\widehat{\mathbf{v}}$ as

$$\widehat{\mathbf{v}}(\mathbf{k}) = r(\mathbf{k}) \exp^{i\varphi(\mathbf{k})} \quad \text{where } r = |\widehat{\mathbf{v}}|, \quad \text{and } \varphi = \arg(\widehat{\mathbf{v}}), \quad (2.564)$$

the phase $\varphi(\mathbf{k})$ is uniformly distributed in $[0, 2\pi]$ and the modulus $r(\mathbf{k})$ follows a Gaussian distribution with zero mean and variance $P(k)$ called the *power spectrum*. A fundamental assumption is that turbulence is isotropic, so that the variance depends only on k but not on the direction of \mathbf{k} . Since the flow is incompressible, the turbulent kinetic energy is defined as

$$K_T = \int_{\mathbb{R}^3} \frac{1}{2} \bar{\rho} v^2 d^3x \quad (2.565)$$

Using Parseval theorem for the Fourier transform and ignoring the constant density $\bar{\rho}$, we get

$$\int_{\mathbb{R}^3} \frac{1}{2} v^2 d^3x = \int_{\mathbb{R}^3} \frac{1}{2} |\widehat{\mathbf{v}}|^2 d^3k = \int_{\mathbb{R}} \frac{1}{2} |\widehat{\mathbf{v}}|^2 4\pi k^2 dk \quad (2.566)$$

For the rightmost equation, we used the isotropy condition. The total turbulent kinetic energy is thus given be

$$\boxed{K_T = \int_{\mathbb{R}} E(k) dk \quad \text{where } E(k) = 2\pi k^2 |\widehat{\mathbf{v}}|^2} \quad (2.567)$$

The function $E(k)$ is called *the energy spectrum of turbulence*.

Kolmogorov scaling relation

Kolmogorov main prediction was that the energy spectrum scales as $k^{-5/3}$, in very good agreement with data. We will derive this result here using simple arguments. We start writing down the viscous Euler equation for an incompressible fluid

$$\frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \nabla P = \nu \Delta \mathbf{v} \quad (2.568)$$

We then Fourier transform it

$$\frac{\partial}{\partial t} \widehat{\mathbf{v}}(\mathbf{k}, t) + \widehat{\mathbf{T}}(\mathbf{k}, t) = -\nu k^2 \widehat{\mathbf{v}}(\mathbf{k}, t) \quad (2.569)$$

and introduce the vector field $\widehat{\mathbf{T}}$ that includes the Fourier transform of the non-linear term and of the pressure gradient. We then multiply by $4\pi k^2 \widehat{\mathbf{v}}(\mathbf{k}, t)$ to make the kinetic energy appear

$$\frac{\partial}{\partial t} E(k, t) + 4\pi k^2 \widehat{\mathbf{v}} \cdot \widehat{\mathbf{T}}(k, t) = -2\nu k^2 E(k, t) \quad (2.570)$$

We now define the *spectral energy flux* so that

$$\frac{\partial}{\partial k} \Pi = 4\pi k^2 \widehat{\mathbf{v}} \cdot \widehat{\mathbf{T}} \quad (2.571)$$

and the energy spectrum equation becomes finally

$$\boxed{\frac{\partial}{\partial t}E(k, t) + \frac{\partial}{\partial k}\Pi(k, t) = -2\nu k^2 E(k, t)} \quad (2.572)$$

The right-hand side stands for viscous dissipation of kinetic energy into heat, while the spectral energy flux describes how kinetic energy flows through scales in Fourier space. The sum over k -space of the viscous term is equal to the dissipation term we have introduced in the mean flow turbulent kinetic energy equation

$$\boxed{\epsilon_T = \int_0^{+\infty} 2\nu k^2 E(k, t) dk} \quad (2.573)$$

We now consider three different scales.

- At small scales, the viscous term will dominate, owing to the k^2 scaling of the Laplace operator. We can then ignore the flux term and solve the heat equation

$$\frac{\partial}{\partial t}E(k, t) = -2\nu k^2 E(k, t) \quad \text{with solution} \quad E(k, t) = E_0(k) \exp^{-2\nu k^2 t} \quad (2.574)$$

We have an exponential suppression of kinetic energy for all scale with $k > k_{\text{diff}} = 1/\sqrt{2\nu t}$. This is the viscous diffusion regime.

- At intermediate scales, we can ignore the viscous effects and the spectral energy equation becomes

$$\frac{\partial}{\partial t}E(k, t) + \frac{\partial}{\partial k}\Pi(k, t) = 0 \quad (2.575)$$

The Kolmogorov spectrum can be obtained if we demand that turbulence reaches a stationary state, for which

$$\frac{\partial}{\partial t}E(k, t) = 0 \quad \text{or equivalently} \quad \Pi = \Pi_0 \quad (2.576)$$

This is called the *inertial regime*.

- At large scales, we need to inject turbulent kinetic energy in the system, otherwise we won't be able to obtain a stationary solution. This injection is due to the production terms that we have derived earlier. They can be modeled by a source term in the previous equation but for only one specific large scale k_{inj}

$$\frac{\partial}{\partial t}E(k, t) + \frac{\partial}{\partial k}\Pi(k, t) = \dot{E}_{\text{inj}}\delta(k - k_{\text{inj}}) \quad (2.577)$$

where δ is the Dirac-Delta function. Note that since for $k < k_{\text{inj}}$, $E = 0$ and $\Pi = 0$, stationarity implies $\Pi(k_{\text{inj}}) = \Pi_0 = \dot{E}_{\text{inj}}$.

We will now model the inertial regime of the turbulent cascade, by writing the energy flux as $\Pi(k) = E(k)\dot{k}$ where \dot{k} is the Fourier space “velocity” of the turbulent cascade. This velocity can be estimated using the so-called *eddy turn-over time* τ_{eddy} , which corresponds to the time it takes for an eddy at scale k to develop secondary instabilities and produce smaller scale vortices. We know from previous sections that this time scale is related to the Kelvin-Helmoltz instability growth rate $\gamma_{\text{KH}} = |V_1 - V_2|k \simeq v(k)k$, where $v(k)$ is the typical velocity of eddies at scale k . Summing up all this, we have

$$\Pi = E\dot{k} = E\frac{k}{\tau_{\text{eddy}}} = Ek\gamma_{\text{KH}} \simeq Evk^2 \quad (2.578)$$

In order to compute $v(k)$, we need to use the energy spectrum and a *high-pass filter*. Indeed, if the total kinetic energy of the fluid is $K_T = \int_0^{+\infty} E(k')dk'$ and corresponds to the kinetic energy of the largest eddies, the kinetic energy associated to eddies of scale k is the integral of the energy spectrum only for $k' > k$.

$$\frac{1}{2}v(k)^2 = \int_k^{+\infty} E(k')dk' = \frac{1}{\alpha-1}E(k)k \quad (2.579)$$

We have assumed that the energy spectrum follows a power law scaling $E(k) \propto k^{-\alpha}$, where $\alpha > 1$ still needs to be determined. We obtain the following form for the flux

$$\Pi = \sqrt{\frac{2}{\alpha-1}}E^{3/2}k^{5/2} = \Pi_0 \quad \rightarrow \quad E(k) = \left(\frac{1}{3}\right)^{1/3} \Pi_0^{2/3}k^{-5/3} \quad (2.580)$$

where we have found and used $\alpha = 5/3$. If we integrate the stationary spectral energy equation between k_{inj} and k_{diff} , we get

$$\int_{k_{\text{inj}}}^{k_{\text{diff}}} \frac{\partial \Pi}{\partial k} = \Pi(k_{\text{diff}}) - \Pi_0 \simeq -\epsilon_T \quad (2.581)$$

At $k \simeq k_{\text{diff}}$, the exponential decay ensures that $E \simeq 0$ and $\Pi \simeq 0$, so we can write $\Pi_0 = \epsilon_T$. The final Kolmogorov spectrum is thus

$$\boxed{E(k) \simeq 0.7 \epsilon_T^{2/3} k^{-5/3}} \quad (2.582)$$

Note that in the Kolmogorov picture, we have $\dot{E}_{\text{inj}} = \Pi_0 = \epsilon_T$, which is also consistent with the mixing length theory and Smagorinsky approximation.

Burgers turbulence and Larson scaling relation

Kolmogorov theory turned out to very successfully explain subsonic turbulence on the Earth for atmospheric and oceanic modelling or engineering applications. What about the interstellar medium? A very famous observational study realized almost 4 decades ago by astrophysicist Richard Larson demonstrated that the 1D velocity dispersion in molecular clouds is strongly correlated with their size and scales as

$$\sigma_v(\ell) \simeq 1 \text{ km/s} \left(\frac{\ell}{1 \text{ pc}}\right)^{1/2} \quad (2.583)$$

Using Kolmogorov theory, we can now compute the typical kinetic energy associated with eddies of size $\ell = 2\pi/k$ as before

$$\frac{1}{2}v(k)^2 = \int_k^{+\infty} E(k')dk' = 3E(k)k \propto k^{-5/3}k \quad \text{so that} \quad \sigma_v(\ell) \propto \ell^{1/3} \quad (2.584)$$

which obviously does not match the observational data. The problem is that Kolmogorov theory only applies to incompressible, subsonic turbulence, and molecular clouds, with a typical velocity of 1 km/s are obviously supersonic since the sound speed of the cold molecular gas is $a \simeq 0.2$ km/s. A better model for supersonic turbulence is given again by the Burgers equation and the formation of shocks. We have discussed already several times the unescapable fact that shock waves form in finite time from initial random planar waves. A better description of the

velocity field, after the shock has formed, is a Heavyside function (see Figure 2.4). In order to simplify the computations, we assume the velocity field is given by

$$v_x(x, y, z) = -1 \quad \text{for } x < 0 \quad \text{and} \quad v_x(x, y, z) = +1 \quad \text{for } x > 0 \quad (2.585)$$

We also see that the velocity field is uniform in the y and z direction. The Fourier transform of such a velocity field is highly anisotropic, and takes the form

$$\widehat{v}_x = -\frac{i}{k_x} \delta(k_y) \delta(k_z) \quad (2.586)$$

where δ is the Dirac-Delta function. The kinetic energy associated to this profile is just

$$K_T = \int_{\mathbb{R}^3} \frac{1}{2} k_x^{-2} \delta(k_y) \delta(k_z) dk_x dk_y dk_z = \int_0^{+\infty} k_x^{-2} dk_x \quad (2.587)$$

For each planar wave, we have a similar energy spectrum. If we have enough random orientations, we can add up all these independant contributions and somewhat restore the isotropy condition. We finally get for Burgers turbulence

$$\boxed{E(k) \propto k^{-2}} \quad (2.588)$$

We can now compute the typical velocity dispersion at scale $\ell = 2\pi/k$

$$\frac{1}{2} v(k)^2 = \int_k^{+\infty} E(k') dk' = E(k) k \propto k^{-2} k \quad \text{so that} \quad \sigma_v(\ell) \propto \ell^{1/2} \quad (2.589)$$

in agreement with Larson relation. The transition between supersonic turbulence and subsonic turbulence occurs at *the sonic scale* ℓ_s defined by $\sigma_v(\ell_s) = a$. Using Larson relation, we find $\ell_s \simeq 0.1$ pc. For $\ell > 0.1$ pc, the interstellar medium turbulence is supersonic, with a lognormal density PDF, while for $\ell < 0.1$ pc, the turbulence is subsonic and the density remains almost constant. Scales smaller than 0.1 pc are associated with *dense molecular cores* which are believed to be the sites of star formation. If a dense subsonic core is gravitationally unstable, it will eventually collapse and form one or two stars.

Chapter 3

Radiative processes in astrophysics

In this chapter, we describe radiative processes in the context of astrophysics. We will introduce the theoretical framework to describe two main processes: 1- the transport of radiation and the equation of radiative transfer, 2- the interaction between matter and radiation that occurs during many different emission and absorption processes. The first half of this chapter will be dedicated to the radiative transfer equation and its various applications for astrophysical fluid flows. The second half will focus on the description of inelastic collisions featuring atoms and molecules, and how they can absorb or emit photons in different regimes.

3.1 Radiative transfer equation

Quantum mechanics is based on the concept of particle-wave duality. Radiation is probably the most extreme quantum object we will be dealing with, owing to its relativistic nature. Radiation can therefore be described as a collection of photons, each one of them being a mass-less boson carrying a quantum of energy $E = h\nu$ and a spin equal to $+1$ or -1 . Radiation can also be described as an electromagnetic wave with frequency $\omega = 2\pi\nu$ following Maxwell's equations with two possible polarisations. We will use both interpretations depending on the context. Since Chapter 1 of this course was dedicated to Boltzmann equation, we will adopt first the photon point of view and exploit what we already know about the phase-space dynamics of a fluid made of zero mass relativistic particles. This will be the basis of our derivation of the radiative transfer equation.

3.1.1 Boltzmann equation for photons

In the kinetic theory framework, we describe photons using phase-space coordinates (\mathbf{q}, \mathbf{p}) . \mathbf{q} is here the spatial coordinate of the fluid element, that we change into the spatial coordinate \mathbf{x} . \mathbf{p} is the photon momentum. Since photons have no mass, we know from the relativistic form of the energy that

$$E^2 = p^2 c^2 + m^2 c^4 \text{ so that } E = pc \text{ or } p = \frac{h\nu}{c} \quad (3.1)$$

Moreover, since the velocity can be obtained from Hamiltonian dynamics using

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}} = c\mathbf{n} \quad (3.2)$$

where \mathbf{n} is the unit vector pointing in the direction of the photon momentum, we conclude that the photons move at the speed of light. The distribution function f is defined using the number of photons per phase-space volume element

$$dN = f(\mathbf{x}, \mathbf{p}, t) d^3x d^3p \quad (3.3)$$

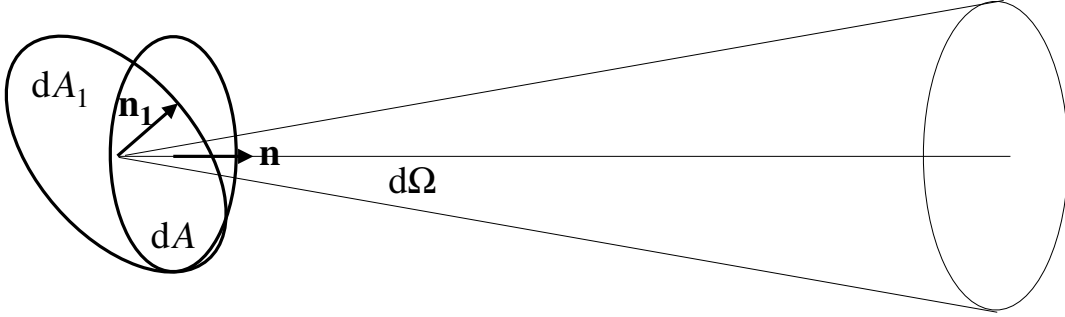


Figure 3.1: Schematics of a light ray impacting a detector surface element dA_1 at an angle θ_1 defined by $\cos \theta_1 = \mathbf{n} \cdot \mathbf{n}_1$.

The resulting Boltzmann equation without collision term is written as (see Chapter 1 for the justification).

$$\frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \quad (3.4)$$

which simplifies in

$$\frac{\partial f}{\partial t} + c\mathbf{n} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0 \quad (3.5)$$

We have used again the fact that photons have no mass and thus cannot feel any Newtonian gravity. The situation is different if one considers general relativity but this is outside the scope of this course. Since the last equation only features the angular variable (θ, ϕ) defining the unit vector \mathbf{n} , it is customary to redefine the distribution function f the following way

$$f(\mathbf{x}, \mathbf{p}, t) = f_\nu(\mathbf{x}, \mathbf{n}, t) \quad (3.6)$$

where the degree of freedom associated to the norm of the momentum vector has been absorbed in the notation using the index ν . This is just a new notation without any deeper meaning.

3.1.2 Radiation specific intensity

Unfortunately, radiation is traditionally not described using the distribution function in phase-space, but using the radiation *specific intensity* I_ν defined as

$$dE = I_\nu(\mathbf{x}, \mathbf{n}, t) dA_1 \cos \theta_1 d\Omega d\nu dt = I_\nu(\mathbf{x}, \mathbf{n}, t) dA d\Omega d\nu dt \quad (3.7)$$

where dE is the energy absorbed by the surface element dA_1 of a detector per units time dt in frequency range $d\nu$, coming from a beam pointing in direction \mathbf{n} and of size $d\Omega$. The angle $\cos \theta_1 = \mathbf{n}_1 \cdot \mathbf{n}$ is the angle between the unit normal vector on the detector \mathbf{n}_1 and the unit vector \mathbf{n} pointing in the direction of the light ray. We can also use the rightmost formulation using the projected surface perpendicular to the light ray $dA = dA_1 \cos \theta_1$. The units of I_ν are $[\text{erg cm}^{-2} \text{rad}^{-1} \text{Hz}^{-1} \text{s}^{-1}]$.

It is straightforward to translate this new, more traditional definition into the kinetic theory distribution function approach using our good-old collision cylinder. Indeed, during time step

dt , the photons along the light ray cover an infinitesimal volume

$$d^3x = c dt dA \quad (3.8)$$

Using the definition of the photon momentum, we know that the momentum space volume element is

$$d^3p = p^2 dp d\Omega = \frac{h^3 \nu^2}{c^3} d\nu d\Omega \quad (3.9)$$

Using the definition of the distribution function, we can compute the energy in the phase-space volume element as

$$dE = h\nu f_\nu d^3x d^3p = \frac{h^4 \nu^3}{c^2} f_\nu dA d\Omega d\nu dt \quad (3.10)$$

Identifying the terms, we finally get for the specific intensity

$$\boxed{I_\nu = \frac{h^4 \nu^3}{c^2} f_\nu} \quad (3.11)$$

3.1.3 Bose-Einstein distribution and the Black Body spectrum

We know from the Chapter on kinetic theory that under local thermodynamical equilibrium (LTE) conditions, the distribution function must satisfy a well defined form. In the case of photons, which are mass-less bosons, we know detailed balance in phase-space leads to a new collision invariant based on the *occupation number* \mathcal{N}

$$\ln \frac{\mathcal{N}}{1 + \mathcal{N}} = -\beta E = -\frac{h\nu}{k_B T} \quad (3.12)$$

In the previous equation, we state that the new collision invariant has to be proportional to the existing ones following the microscopic conservation laws. Since photons have no mass, we do not have the coefficient related to mass conservation, namely the chemical potential $\mu(\mathbf{x}, t) = 0$. We only have the multiplier associated to the total energy, which is noted here $\beta(\mathbf{x}, t)^{-1} = k_B T(\mathbf{x}, t)$. We also assume that we are in the *comoving frame* of the fluid, so that there is no term associated to the fluid bulk velocity. These terms will appear once we perform a Lorentz transform from the comoving frame to the laboratory frame. In case of radiation, this comoving frame versus laboratory frame distinction is of crucial importance. We will first ignore it, neglecting relativistic effects, but we will have to introduce them later to get a consistent description of radiation hydrodynamics.

Solving the previous equation for \mathcal{N} , and using the relation between the distribution function and the occupation number, we obtain *the Bose-Einstein* distribution function

$$f_\nu = \frac{2}{h^3} \mathcal{N} = \frac{2}{h^3} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} \quad (3.13)$$

where we have multiplied the occupation number by 2 to account for the two possible spin states. Finally, using the previously derived relation between the specific radiation intensity and the distribution function, we deduce the LTE specific intensity called *the black body spectrum* as

$$\boxed{B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}} \quad (3.14)$$

Note that for radiation, LTE is not reached through photon-photon collisions, but rather through emission and absorption processes due to matter and radiation interactions. We will see how

it works in details later. This radiation spectrum depends only on the frequency ν and on the temperature T . Note that it does not depend on the angle variables, it is therefore an isotropic distribution function, like the Maxwell-Boltzmann distribution for ideal gases. This spectrum reaches its maximum at the wavelength

$$\lambda_{\max} = \frac{c}{\nu_{\max}} = \frac{0.3}{T} \text{ cm} \quad (3.15)$$

where T is expressed in Kelvin. The surface temperature of the Sun is around $T \simeq 6000$ K, so that its radiation spectrum peaks at $\lambda_{\max} \simeq 500$ nm, giving it its yellow colour. For frequencies higher than the peak frequency, the specific intensity is decreasing exponentially fast towards zero. This is known as Wien's law. For smaller frequency $\nu \ll \nu_{\max}$, we are in the Rayleigh-Jeans regime, and the black body radiation is proportional to the temperature with

$$B_{\nu}(T) \simeq \frac{2\nu^2}{c^2} k_{\text{B}}T \quad (3.16)$$

Even if the radiation specific intensity is not a perfect black body spectrum, because the system is not in perfect LTE, it is customary to define the *brightness temperature* of the radiation as

$$k_{\text{B}}T_{\text{B}} = \frac{c^2}{2\nu^2} I_{\nu} \quad (3.17)$$

The brightness temperature corresponds to a true temperature if and only if the radiation is at LTE.

3.1.4 Radiative transfer equation in vacuum

We will now translate the Boltzmann equation for the radiation using the previously defined specific intensity. Since \mathbf{x} , t and ν are all independent variables, we can just re-write the Boltzmann equation as

$$\frac{1}{c} \frac{\partial I_{\nu}}{\partial t} + \mathbf{n} \cdot \nabla I_{\nu} = 0 \quad (3.18)$$

Note that we have just replaced f_{ν} by I_{ν} and divide the entire equation by c . This is a convention for the radiative transfer equation that we explain now. The Boltzmann equation can be interpreted as a Lagrange derivative of the specific intensity along the trajectory of the light ray, moving at the speed of light in direction \mathbf{n} . The specific intensity is therefore conserved along straight lines when the radiation propagates in vacuum. We can also define along the same light ray the curvilinear coordinate s as $ds = cdt$. The radiative transfer equation in vacuum is therefore equivalent to

$$\frac{dI_{\nu}}{ds} = 0 \quad (3.19)$$

Note that the curvilinear coordinate is unique to each light ray. In order to describe the radiation field in a full 3D sense, one must use the first form of the radiative transfer equation with coordinates \mathbf{x} , \mathbf{n} and t .

3.1.5 Interaction between matter and radiation

Most collisions in the fluid are elastic collisions. They conserve strictly mass, momentum and energy at the microscopic level. A few collisions are however inelastic, leading to the emission or the absorption of a photon. For example, an hydrogen atom with its electron on the most bound quantum state, called the fundamental level, can collide with a free electron, its most probable collision partner. Most of the energy will be absorbed by the new kinetic energies of

the collision partners, but some of the energy can be spent in exciting the bound electron from the fundamental level to an excited state. This is an inelastic collision because the incoming particle kinetic energy is decomposed into excitation energy for the bound electron and kinetic energy for the outgoing particle. This excited state will later spontaneously decay back into the fundamental level, emitting a photon in the process. This leads to a net production of radiation emerging from the matter. The reverse situation is also possible, with an incoming photon absorbed by an atom, leading to the ejection of the bound electron into the population of free electron. This process is called photo-ionisation and leads to a net absorption of radiation by the matter. These 2 processes are just 2 examples of radiation and matter interaction, either the emission of radiation by matter or the absorption of radiation by matter. We will describe in great details all the classical processes that are relevant for astrophysical radiative fluids. These processes are described using j_ν , the emission coefficient and α_ν , the absorption coefficient, both present in the right-hand side of the radiative transfer equation as

$$\boxed{\frac{1}{c} \frac{\partial I_\nu}{\partial t} + \mathbf{n} \cdot \nabla I_\nu = j_\nu - \alpha_\nu I_\nu} \quad (3.20)$$

These two new terms are the equivalent of the collision integral in the Boltzmann equation. Note that the absorption term is proportional to the incoming radiation specific intensity. The units of j_ν are $[\text{erg cm}^{-3} \text{rad}^{-1} \text{Hz}^{-1} \text{s}^{-1}]$, while the units of α_ν are simply $[\text{cm}^{-1}]$. The emissivity coefficient is traditionally decomposed into spontaneous emission and induced (or stimulated) emission, as

$$j_\nu = j_\nu^s + j_\nu^i = j_\nu^s + \alpha_\nu^i I_\nu \quad (3.21)$$

where we used the fact that the induced emissivity coefficient is proportional to the incoming radiation specific intensity. Stimulated emission is due to the friendly nature of bosons who tend to gather together, with an enhanced probability of emission in phase-space volume elements already occupied by other bosons. We have described this effect using the Bose enhancement factor $1 + \mathcal{N}$. Here also, we have to multiply the emissivity coefficient by this Bose enhancement factor, leading to the induced emission term. The induced emission term can in fact be absorbed into the absorption coefficient, with a negative sign, leaving the original form of the radiative transfer equation unchanged, just corrected from induced (or stimulated) emission.

The absorption coefficient α_ν can be also defined using either the mean free path λ_ν , the opacity κ_ν or the cross-section σ_ν by

$$\alpha_\nu = \frac{1}{\lambda_\nu} = \rho \kappa_\nu = n \sigma_\nu \quad (3.22)$$

where λ_ν is in [cm] and κ_ν is in $[\text{cm}^2 \text{g}^{-1}]$. The opacity can be understood in term of cross-section per particle of fluid. A particle can indeed be considered as a small sphere of radius a with a apparent cross-section $\sigma_\nu = \pi a^2$ blocking the incoming light. The opacity is then defined as $\sigma_\nu = m \kappa_\nu$ where m is the mass of the particle.

3.2 Formal solution of the radiative transfer equation

In this section, we will solve the radiative transfer equation in a formal sense. This means we will have a full mathematical expression describing the evolution of the specific intensity along the light ray, but this expression is only of theoretical interest. We will however consider some simplified cases where analytical expressions can be found and new insight on the propagation of radiation in a gaseous medium can be gained.

3.2.1 Optical depth

We start with the radiative transfer equation in Lagrangian form using the curvilinear coordinate s along the light ray.

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu \quad (3.23)$$

Dividing by α_ν , we obtain the following useful form

$$\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu \quad (3.24)$$

where we have introduced the *optical depth* τ_ν , a dimensionless quantity defined by

$$d\tau_\nu = \alpha_\nu ds \text{ or } \tau_\nu(s) = \int_0^s \alpha_\nu(s') ds' \quad (3.25)$$

3.2.2 Source function and Kirchhoff's law

We see in the previous equation that the right-hand side is the balance between a source term and a sink term. In the frame of the fluid, or in the frame of the laboratory if the fluid is at rest, one expects the source term to be isotropic. Indeed, if the fluid is in LTE, particles follow a Maxwell-Boltzmann distribution in velocity space, and photons emitted by atoms and molecules must come with equal probability in all directions. Note that if the fluid is moving, the Doppler shift due to the bulk motion of the fluid will introduce a small anisotropy of the emission in the laboratory frame, not in the comoving frame. In what follows, we ignore this relativistic effect and assume that the source term, called *the source function*, is always isotropic in both frames. The source function is noted S_ν and is equal to

$$S_\nu(T_{\text{gas}}) = \frac{j_\nu}{\alpha_\nu} \quad (3.26)$$

Note that it depends on the gas temperature. If the gas is at LTE, it does not depend on the radiation field. If now, we consider that radiation is also at LTE, then we know that the specific intensity must follow the black body spectrum with the radiation temperature $B_\nu(T_{\text{rad}})$. In the case of full LTE, for which $T_{\text{gas}} = T_{\text{rad}} = T$, the sink term must balance exactly the source term and one has $S_\nu(T) = B_\nu(T)$. The final argument is as follows: since the source function depends only on gas properties (if this one is at LTE), then it must always satisfy $S_\nu(T_{\text{gas}}) = B_\nu(T_{\text{gas}})$, even if the radiation is not at LTE. This property is known as Kirchhoff's law and writes

$$\frac{j_\nu}{\alpha_\nu} = B_\nu(T_{\text{gas}}) \quad (3.27)$$

This is the regime of *thermal radiation*.

3.2.3 Formal solution along the light ray

We now consider the general case where neither radiation nor gas are in LTE. We work on a light ray that enters the slab at $s = 0$ and leaves the slab at $s = S$, so that S is the thickness of the slab, as seen by the light ray. The light ray is coming from the left, say. We assume it originates from a background star on the far left and follows a black body spectrum with temperature T_* . Because the region on the left is in vacuum, one has $I_\nu(s = 0) = B_\nu(T_*)$. We would like to compute the evolution of the radiation specific intensity along the light ray, and

especially how it emerges from the slab and appears to an observer located on the far right of the slab. We use the radiative transfer equation in its optical depth formulation as

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \quad (3.28)$$

where τ_ν is the integration variable. This is a first order linear ordinary differential equation. We apply the usual method, solving first the homogeneous equation

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu \longrightarrow I_\nu(\tau_\nu) = I_\nu(0) \exp^{-\tau_\nu} = B_\nu(T_*) \exp^{-\tau_\nu} \quad (3.29)$$

where, owing to the definition of τ_ν , we have $\tau_\nu = 0$ at $s = 0$. This homogeneous solution corresponds to the attenuation of the star radiation by the absorbing layer of gas. In this case, the observed radiation on the far right is just

$$I_\nu^{\text{obs}} = B_\nu(T_*) \exp^{-T_\nu} \quad (3.30)$$

where T_ν is the total optical depth of the cloud calculated as

$$T_\nu = \int_0^S \alpha_\nu(s') ds' \quad (3.31)$$

In order to find the general solution, we use the technique of the variation of the constant with the Ansatz

$$I_\nu = J_\nu \exp^{-\tau_\nu} \quad (3.32)$$

Injecting this Ansatz into the original differential equation, we obtain

$$\frac{dJ_\nu}{d\tau_\nu} = S_\nu \exp^{+\tau_\nu} \longrightarrow J_\nu = \int_0^{\tau_\nu} S_\nu \exp^{+\tau'_\nu} d\tau'_\nu + B_\nu(T_*) \quad (3.33)$$

Note that we introduced τ'_ν as dummy integration variable. We finally get the complete and final formal solution to the radiative transfer problem along this single light ray

$$I_\nu(\tau_\nu) = \int_0^{\tau_\nu} S_\nu(\tau'_\nu) \exp^{(\tau'_\nu - \tau_\nu)} d\tau'_\nu + B_\nu(T_*) \exp^{-\tau_\nu} \quad (3.34)$$

Let's consider the particular case for which the source function is uniform throughout the gaseous slab $S_\nu = B_\nu(T_{\text{gas}})$ where T_{gas} is uniform. we can trivially integrate the previous equation and obtain

$$I_\nu(\tau_\nu) = B_\nu(T_{\text{gas}}) (1 - \exp^{-\tau_\nu}) + B_\nu(T_*) \exp^{-\tau_\nu} \quad (3.35)$$

We have two limiting cases:

- $\tau_\nu \gg 1$: this is the optically thick regime, for which the specific intensity converges exponentially fast towards the black body spectrum at the gas temperature. Since this spectrum is isotropic, we conclude that in the optically thick regime, the radiation is driven towards LTE, with both an isotropic angular distribution, independent on the chosen light ray, and a black body spectrum with $T_{\text{rad}} = T_{\text{gas}}$.
- $\tau_\nu \ll 1$: this is the optically thin regime. In this case, we can Taylor expand the two exponentials and obtain $I_\nu(\tau_\nu) \simeq B_\nu(T_*) + (B_\nu(T_{\text{gas}}) - B_\nu(T_*)) \tau_\nu$. If $T_{\text{gas}} > T_*$, the radiation intensity in the slab is larger than the background stellar light. The radiation is said to be *in emission*. In the opposite case, the stellar light is attenuated and the radiation is said to be *in absorption*. Interestingly, the emergent optically thin radiation is directly proportional to τ_ν and can be used to infer physical properties of the absorbing gas.

If the optical depth of the cloud is very large $T_\nu \gg 1$, we can compute the emerging radiation as seen from the observer as

$$I_\nu(T_\nu) \simeq \int_0^{T_\nu} S_\nu(\tau'_\nu) \exp(\tau'_\nu - T_\nu) d\tau'_\nu \quad (3.36)$$

The background stellar radiation has been totally absorbed by the gaseous slab. We can again change variable and define $\tau'_\nu = 0$ on the far right end of the slab, with $\tau'_\nu \rightarrow T_\nu - \tau'_\nu$. We get

$$I_\nu^{\text{obs}} \simeq \int_0^{T_\nu} S_\nu(\tau'_\nu) \exp^{-\tau'_\nu} d\tau'_\nu \simeq \int_0^{+\infty} S_\nu(\tau'_\nu) \exp^{-\tau'_\nu} d\tau'_\nu \quad (3.37)$$

Decomposing the source function into a first order Taylor expansion,

$$S_\nu(\tau_\nu) \simeq S_\nu^{(0)+} S_\nu^{(1)}(0) \tau_\nu \quad (3.38)$$

we find the *Eddington-Barbier relation* which states that

$$I_\nu^{\text{obs}} \simeq S_\nu(\tau_\nu = 1) \quad (3.39)$$

This relation, surprisingly accurate in optically thick conditions, states that the emergent radiation spectrum is equal to the source function evaluated at a position along the light ray precisely defined at $\tau_\nu = 1$. In stars, this position is called the *photo-sphere*. This relation is also at the origin of the Limb darkening of the radiation emitted by the solar surface (left to the reader as an exercise).

3.3 Moments of the radiative transfer equation

So far, we have described the evolution of radiation in a fluid only for one direction, namely along one light ray, and for one frequency. Solving for the full problem requires to repeat this for all directions and for all frequencies, a daunting task reminiscent of the challenges of solving the full Boltzmann equation in kinetic theory. Fortunately, following the strategy we have presented in Chapter 1, we can simplify the problem by integrating out the angular and frequency variables. We have called this operation “taking moments” of the distribution function or “taking moments” of the Boltzmann equation. Here again, we will do the same and take the moments of both the specific intensity and the radiative transfer equation.

3.3.1 Moments of the specific intensity

We first define the radiation energy density u_ν using the same approach of the previous section, where we have connected the specific intensity to the photon distribution function. Since the volume element covered by radiation flowing at the speed of light is $d^3x = c dt dA$, we can write the energy in this volume element using both definitions as

$$dE = u_\nu d^3x d\nu d\Omega = u_\nu c dt dA d\nu d\Omega = I_\nu dt dA d\nu d\Omega \quad (3.40)$$

We deduce that the radiation energy density, expressed in units of $[\text{erg cm}^{-3} \text{Hz}^{-1} \text{rad}^{-1}]$ is then

$$u_\nu(\mathbf{x}, \mathbf{n}, t) = \frac{I_\nu}{c} \quad (3.41)$$

The first moment is defined as *the total energy density* and is the integral of the previous quantity over the solid angle.

$$E_\nu(\mathbf{x}, t) = \int_{4\pi} \frac{I_\nu}{c} d\Omega \quad (3.42)$$

This new quantity depends only on the space and time coordinates. Its units are $[\text{erg cm}^{-3}\text{Hz}^{-1}]$. It is customary to define also the *mean radiation specific intensity* as

$$J_\nu(\mathbf{x}, t) = \int_{4\pi} I_\nu \frac{d\Omega}{4\pi} = \frac{c}{4\pi} E_\nu \quad (3.43)$$

We can finally integrate the total energy density over frequencies and obtain the total integrated radiation energy density, in units of $[\text{erg cm}^{-3}]$

$$E_{\text{rad}}(\mathbf{x}, t) = \int_0^{+\infty} E_\nu d\nu \quad (3.44)$$

where we used index “rad” to avoid confusion with the fluid total energy. If the radiation is in LTE, we have $I_\nu = B_\nu(T)$ and strictly isotropic. The total energy radiation density is then

$$E_\nu(\mathbf{x}, t) = \frac{4\pi}{c} B_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} \quad (3.45)$$

We can easily integrate this over frequencies and get the black body integrated energy as

$$E_{\text{rad}}(T) = \frac{8\pi h}{c^3} \int_0^{+\infty} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} d\nu = \frac{8\pi h}{c^3} \left(\frac{k_B T}{h}\right)^4 \int_0^{+\infty} \frac{x^3}{\exp^x - 1} dx \quad (3.46)$$

Since everyone knows that

$$\int_0^{+\infty} \frac{x^3}{\exp^x - 1} dx = \frac{\pi^4}{15} \quad (3.47)$$

we finally get for the black body radiation

$$E_{\text{rad}}(T) = aT^4 \text{ where Stefan's constant is } a = \frac{8\pi^5}{15} \frac{k_B^4}{h^3 c^3} \quad (3.48)$$

The striking feature of the black body is that the radiation energy depends only on temperature, as opposed to a Maxwell-Boltzmann gas, for which the internal energy depends also on density, or even only on density if it is degenerate. This is a consequence of photons having zero mass.

In analogy to kinetic theory, we now define more moments of the specific intensity that will prove useful later. The radiation flux is defined as a vector field

$$\mathbf{F}_\nu(\mathbf{x}, t) = \int_{4\pi} I_\nu \mathbf{n} d\Omega \quad (3.49)$$

While the radiation energy density is considered as a zeroth order moment of the specific intensity, the radiation flux is the first order moment, because it comes with a polynomial of \mathbf{n} of degree 1. If the radiation field is isotropic, like black body radiation, I_ν can be taken out of the integral and the flux is trivially zero. Note that we can also define the momentum density of the radiation by noticing that the momentum of a photon is given by its amplitude

$$p = \frac{h\nu}{c} \text{ so that } p_\nu = \frac{u_\nu}{c} = \frac{I_\nu}{c^2} \text{ and } \mathbf{p}_\nu = \frac{I_\nu}{c^2} \mathbf{n} \quad (3.50)$$

Finally, integrating over the solid angle, we get the total radiation momentum density as

$$\mathbf{P}_\nu(\mathbf{x}, t) = \frac{\mathbf{F}_\nu}{c^2} \quad (3.51)$$

This is the radiation equivalent to the fluid momentum $\rho\mathbf{v}$. We can follow the analogy with a fluid even more and define the radiation pressure tensor as the following second-order moment over the solid angle

$$\mathbb{P}_\nu = \int_{4\pi} \frac{I_\nu}{c} \mathbf{n} \otimes \mathbf{n} d\Omega \quad (3.52)$$

Note that the pressure tensor has the property

$$\text{Tr } \mathbb{P}_\nu = \int_{4\pi} \frac{I_\nu}{c} d\Omega = E_\nu \quad (3.53)$$

In case of isotropic black body radiation, the tensor is also isotropic and can be written as

$$\mathbb{P}_\nu = P_\nu \mathbb{I} \text{ where } P_\nu = \frac{1}{3} E_\nu \quad (3.54)$$

The scalar quantity P_ν is called the radiation pressure, and make sense only under optically thick conditions. Note that we find that isotropic radiation behaves like a $\gamma = 4/3$ fluid, similar to a relativistic Maxwell-Boltzmann gas.

3.3.2 Radiation energy conservation equation

We now move to the task of taking the moments of the radiative transfer equation, integrating it over the solid angles. We first write the radiative transfer equation using time, space and angle coordinates

$$\frac{1}{c} \frac{\partial I_\nu}{\partial t} + \mathbf{n} \cdot \nabla I_\nu = j_\nu - \alpha_\nu I_\nu \quad (3.55)$$

We then perform the integration over 4π , bearing in mind that t , \mathbf{x} and \mathbf{n} are independent variables.

$$\int_{4\pi} \frac{1}{c} \frac{\partial I_\nu}{\partial t} d\Omega + \int_{4\pi} \mathbf{n} \cdot \nabla I_\nu d\Omega = \int_{4\pi} j_\nu d\Omega - \int_{4\pi} \alpha_\nu I_\nu d\Omega \quad (3.56)$$

(1)
(2)
(3)
(4)

The first term, labelled (1) can be computed by taking the time derivative out of the integral, owing to the independence of t and \mathbf{n} . This reveals the time derivative of the total radiation energy density as

$$\frac{\partial}{\partial t} \int_{4\pi} \frac{I_\nu}{c} d\Omega = \frac{\partial E_\nu}{\partial t} \quad (3.57)$$

The second term can be computed using the famous vector relation

$$\mathbf{n} \cdot \nabla I_\nu = \nabla \cdot (I_\nu \mathbf{n}) - I_\nu \nabla \cdot \mathbf{n} \quad (3.58)$$

Since $\nabla \cdot \mathbf{n} = 0$, owing to the independence of \mathbf{x} and \mathbf{n} , we can take the divergence operator out of the integral as

$$\nabla \cdot \left[\int_{4\pi} I_\nu \mathbf{n} d\Omega \right] \equiv \nabla \cdot \mathbf{F}_\nu \quad (3.59)$$

where we recognise the previously defined radiation flux

$$\mathbf{F}_\nu = \int_{4\pi} I_\nu \mathbf{n} d\Omega \quad (3.60)$$

We now deal with the source and sink terms on the right-hand side. The emissivity and absorption coefficients j_ν and α_ν are considered here as isotropic in the laboratory frame. This is in

fact true only in the comoving frame, but we will correct for the relativistic Doppler effect in the following sections. They can thus be taken out of the integral, leading to the final form of the radiation energy conservation equation

$$\boxed{\frac{\partial E_\nu}{\partial t} + \nabla \cdot \mathbf{F}_\nu = 4\pi j_\nu - \alpha_\nu c E_\nu} \quad (3.61)$$

3.3.3 Radiation flux conservation equation

We then follow the same procedure, this time multiplying the radiative transfer equation by n_i , each component of the direction vector \mathbf{n} .

$$\int_{4\pi} n_i \frac{1}{c} \frac{\partial I_\nu}{\partial t} d\Omega + \int_{4\pi} n_i \mathbf{n} \cdot \nabla I_\nu d\Omega - \int_{4\pi} n_i j_\nu d\Omega - \int_{4\pi} n_i \alpha_\nu I_\nu d\Omega \quad (3.62)$$

(1) (2) (3) (4)

The first term can be written as the time derivative of the first component of the radiation flux

$$(1) = \frac{1}{c} \frac{\partial F_i}{\partial t} \quad (3.63)$$

The second term, using the same vector relation as before, gives

$$(2) = \nabla \cdot \left(\int_{4\pi} I_\nu n_i \mathbf{n} d\Omega \right) \quad (3.64)$$

We recognise the first row of the previously defined radiation pressure tensor

$$\mathbb{P}_\nu = \int_{4\pi} \mathbf{n} \otimes \mathbf{n} \frac{I_\nu}{c} d\Omega \quad (3.65)$$

The remaining terms of the right-hand side are respectively (3) zero, owing to the isotropy of the emissivity coefficient, and (4) proportional to the radiation flux. We finally get in full vector and tensor notations

$$\boxed{\frac{1}{c} \frac{\partial \mathbf{F}_\nu}{\partial t} + c \nabla \cdot \mathbb{P}_\nu = -\alpha_\nu \mathbf{F}_\nu} \quad (3.66)$$

3.3.4 Eddington tensor and closure relations

In the context of radiation transfer, it is customary to define the *Eddington tensor* as

$$\mathbb{P}_\nu = \mathbb{D}_\nu E_\nu \quad (3.67)$$

The Eddington tensor is a dimensionless tensor that fully encodes the geometry of the radiation field and its effect in the radiation flux conservation equation. In order to solve the previous system of equations, we need a model for the Eddington tensor. There are various approximations one can use to close the otherwise infinite moments hierarchy. If we adopt a model for the Eddington tensor, the moment hierarchy is closed at second-order.

The first and simplest model is to assume that the radiation is isotropic in angular space, like for example in the black body case, for which we have

$$\mathbb{D}_\nu = \frac{1}{3} \mathbb{I} \text{ and } \mathbb{P}_\nu = \frac{1}{3} E_\nu \mathbb{I} \quad (3.68)$$

Note that the radiation is not isotropic, we only assume the Eddington tensor is isotropic, which is obviously a very rough approximation, except in the optically thick limit.

The second simple case is when we have only one bright source, say a star, in the optically thin limit. At each point in space, we have

$$I_\nu(\mathbf{n}) = I_\nu^* \delta(\mathbf{n} - \mathbf{n}_0) \quad (3.69)$$

where \mathbf{n}_0 is the unit vector in the direction of the source. In this limit, the total radiation energy is just

$$E_\nu = \int_{4\pi} \frac{I_\nu}{c} d\Omega = \frac{I_\nu^*}{c} \quad (3.70)$$

and the radiation flux is

$$\mathbf{F}_\nu = \int_{4\pi} I_\nu \mathbf{n} d\Omega = I_\nu^* \mathbf{n}_0 = cE_\nu \mathbf{n}_0 \quad (3.71)$$

In the optically thin limit, with only one source, the radiation flux is just the radiation energy times the speed of light in the direction of the source. We can also compute the Eddington tensor in the same regime,

$$\mathbb{P}_\nu = \int_{4\pi} \frac{I_\nu}{c} \mathbf{n} \otimes \mathbf{n} d\Omega = \frac{I_\nu^*}{c} \mathbf{n}_0 \otimes \mathbf{n}_0 = E_\nu \mathbf{n}_0 \otimes \mathbf{n}_0 \quad (3.72)$$

A third slightly more complicated model, called the M1 closure, was proposed by Levermore (1984) and interpolate the previous two regime based on the magnitude of the flux F_ν compared to its maximum allowed value of cE_ν . The M1 closure uses the following form for the Eddington tensor

$$\mathbb{D}_\nu = \frac{1 - \chi}{2} \mathbf{I} + \frac{3\chi - 1}{2} \mathbf{n} \otimes \mathbf{n} \quad (3.73)$$

where $\mathbf{n} = \mathbf{F}_\nu / F_\nu$ and the function $\chi(f)$, not shown here, depends on $f = F_\nu / (cE_\nu)$, a quantity called the *reduced flux*, and reproduces the two previously discussed asymptotic cases $\chi = 1/3$ (isotropic) for $f = 0$ and $\chi = 1$ (single optically thin source) for $f = 1$

More complex closure models are possible. The Optically Thin Variable Eddington Tensor model, for example, assume that the Eddington tensor comes from the collective radiation of a collection of N sources in an optically thin medium. The corresponding radiation field is used only for the computation of the Eddington tensor, and writes

$$I_\nu(\mathbf{n}) = \sum_{i=1}^N I_\nu^i \delta(\mathbf{n} - \mathbf{n}_i) \quad (3.74)$$

Another method, called the Stationary Variable Eddington Tensor model, solves first for the stationary radiative transfer equation

$$\mathbf{n} \cdot \nabla I_\nu = j_\nu - \alpha_\nu I_\nu \quad (3.75)$$

without the time derivative, and then compute the corresponding Eddington tensor. These two methods require a proper numerical integration scheme and quite expensive calculations, but are still simpler and cheaper than solving the full time dependent radiative transfer equation.

3.4 Radiation hydrodynamics

We can now integrate the two previous moments equations over the frequency. We obtain the frequency integrated radiation energy equation as

$$\frac{\partial E_{\text{rad}}}{\partial t} + \nabla \cdot \mathbf{F}_{\text{rad}} = 4\pi \int_0^{+\infty} j_\nu d\nu - \int_0^{+\infty} \alpha_\nu c E_\nu d\nu \quad (3.76)$$

The first term on the right-hand side is called the *cooling function*, and it represents the emission of radiation from matter. It is noted here \mathcal{C}_{rad} and is given by

$$\boxed{\mathcal{C}_{\text{rad}} = 4\pi \int_0^{+\infty} j_\nu d\nu} \quad (3.77)$$

If and only if the gas is in LTE (regime of thermal radiation), one can write the cooling function as

$$\mathcal{C}_{\text{rad}} = 4\pi \int_0^{+\infty} \alpha_\nu B_\nu(T_{\text{gas}}) d\nu = \alpha_{\text{P}} c a T_{\text{gas}}^4 \quad (3.78)$$

where we have defined the *Planck mean* for the absorption coefficient α_{P} as

$$\alpha_{\text{P}}(T_{\text{gas}}) = \frac{\int_0^{+\infty} \alpha_\nu B_\nu(T_{\text{gas}}) d\nu}{\int_0^{+\infty} B_\nu(T_{\text{gas}}) d\nu} \quad (3.79)$$

The second term on the right-hand side is called the *heating function*. It represents the radiation absorbed by matter. It is noted \mathcal{H}_{rad} and is given by

$$\boxed{\mathcal{H}_{\text{rad}} = \int_0^{+\infty} \alpha_\nu c E_\nu d\nu = \alpha_{\text{E}} c E_{\text{rad}}} \quad (3.80)$$

where we have defined the *energy mean* for the absorption coefficient α_{E} as

$$\alpha_{\text{E}} = \frac{\int_0^{+\infty} \alpha_\nu E_\nu d\nu}{\int_0^{+\infty} E_\nu d\nu} \quad (3.81)$$

If and only if the radiation is in LTE (black body regime), one has $E_{\text{rad}} = aT_{\text{rad}}^4$ and $\alpha_{\text{E}} = \alpha_{\text{P}}(T_{\text{rad}})$. Obviously, if $T_{\text{rad}} = T_{\text{gas}}$, we are in full LTE and the two terms exactly cancel out. Integrating the radiation flux conservation equation over frequency, we get

$$\frac{1}{c} \frac{\partial \mathbf{F}_{\text{rad}}}{\partial t} + c \nabla \cdot \mathbb{P}_{\text{rad}} = - \int_0^{+\infty} \alpha_\nu \mathbf{F}_\nu d\nu = -\alpha_{\text{F}} \mathbf{F}_{\text{rad}} \quad (3.82)$$

where we have defined the *flux mean* for the absorption coefficient α_{F} as

$$\alpha_{\text{F}} = \frac{\int_0^{+\infty} \alpha_\nu F_\nu d\nu}{\int_0^{+\infty} F_\nu d\nu} \quad (3.83)$$

Dividing by c , we can write the radiation momentum conservation equation as

$$\boxed{\frac{1}{c^2} \frac{\partial \mathbf{F}_{\text{rad}}}{\partial t} + \nabla \cdot \mathbb{P}_{\text{rad}} = - \int_0^{+\infty} \frac{\alpha_\nu \mathbf{F}_\nu}{c} d\nu = -\frac{\alpha_{\text{F}} \mathbf{F}_{\text{rad}}}{c}} \quad (3.84)$$

Note that the radiation energy emitted by the gas must correspond to thermal energy lost by the gas during inelastic collisions. The same applies for the momentum lost by radiation: It

must correspond to a net gain of momentum by the gas. We therefore have to modify the gas momentum and energy equation to account for the total energy and momentum balance. We therefore have modified Euler equations for the gas. We start with the momentum conservation equation which now reads

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P = \rho \mathbf{a} + \rho \mathbf{a}_{\text{rad}}}$$
 (3.85)

where the *radiative force* (or the radiative acceleration) reads

$$\boxed{\rho \mathbf{a}_{\text{rad}} = \int_0^{+\infty} \frac{\alpha_\nu \mathbf{F}_\nu}{c} d\nu = \frac{\alpha_F \mathbf{F}_{\text{rad}}}{c}}$$
 (3.86)

For the fluid total energy equation, we have to account for two effects: First, we now have a new external force, in addition to gravity, namely the radiative force. The modified momentum conservation equation, when combined with the unmodified mass conservation, will lead to an additional term in the energy equation due to the work of the radiative force. Second, we have also to add source and sink of energy corresponding to the cooling and heating functions.

$$\boxed{\frac{\partial E}{\partial t} + \nabla \cdot (E + P) \mathbf{v} = \rho \mathbf{a} \cdot \mathbf{v} + \rho \mathbf{a}_{\text{rad}} \cdot \mathbf{v} + \mathcal{H}_{\text{rad}} - \mathcal{C}_{\text{rad}}}$$
 (3.87)

We *at last* derived the cooling and heating terms that we used multiple times in the Chapter on astrophysical fluid dynamics. We see that they both originates from radiation and matter interaction. The radiative force, however, is a new concept that we have not introduced earlier. It plays a very important role in many astrophysical objects where the radiation flux is particularly intense, mostly around massive stars or accretion disk around black holes. We will see later that it also plays a fundamental role when the fluid is close to LTE, in the so-called diffusion limit.

These 4 equations define the *radiation hydrodynamics* system of equations. It features conservation of mass, only for the fluid, conservation of momentum for both radiation and fluid, and conservation of energy for both radiation and fluid. Note that we can also add up the two momentum (and energy) conservation equations and obtain new conservation laws for the full system radiation + fluid.

3.5 Diffusion limit

We have seen in the previous section that under optically thick conditions, the specific intensity converges towards the black body spectrum. The resulting isotropic angular distribution of the radiation leads to a vanishing radiation flux. This corresponds to strict LTE conditions for the radiation. Like for kinetic theory, we would like to study the regime when radiation is close to LTE but not quite there yet, so that the radiation field is close to a black body, but not strictly isotropic. We expect a small but non zero radiation flux. It is in fact possible to compute the flux in that regime, following a methodology close to the Chapman-Enskog derivation for viscosity in non-ideal fluids.

We assume that the radiation specific intensity is close to a black body but not strictly equal, so that we have a small perturbation

$$I_\nu(\mathbf{x}, \mathbf{n}, t) = B_\nu(T) + \delta I_\nu(\mathbf{n}) \text{ where } \delta I_\nu \ll I_\nu$$
 (3.88)

The resulting total radiation energy can be written as

$$E_\nu(\mathbf{x}, t) = \frac{4\pi}{c} B_\nu(T) + \delta E_\nu(\mathbf{x}, t) \text{ where } \delta E_\nu \ll E_\nu$$
 (3.89)

and the radiation flux as

$$\mathbf{F}_\nu(\mathbf{x}, t) = 0 + \delta\mathbf{F}_\nu(\mathbf{x}, t) \text{ where } \delta F_\nu \ll cE_\nu \quad (3.90)$$

We now follow the Chapman-Enskog methodology, introducing the reduced flux, so that one can write

$$\mathbf{F}_\nu(\mathbf{x}, t) = \mathbf{f}_\nu cE_\nu \text{ where } f_\nu \ll 1 \quad (3.91)$$

We now inject this form into the radiation flux conservation equation

$$\frac{1}{c} \frac{\partial}{\partial t} (\mathbf{f}_\nu cE_\nu) + \nabla \cdot (\mathbb{D}_\nu cE_\nu) = -\alpha_\nu \mathbf{f}_\nu cE_\nu \quad (3.92)$$

where we have used the Eddington tensor form of the radiation pressure tensor. We now perform an order of magnitude analysis of the previous equation, introducing the typical system size L , the typical fluid time-scale T and the typical fluid velocity $V = L/T$. The previous equation can be written as an order of magnitude estimate

$$\left[\frac{f_\nu cE_\nu}{cT} \right] + \left[\frac{D_\nu cE_\nu}{L} \right] = - \left[\frac{f_\nu cE_\nu}{\lambda_\nu} \right] \quad (3.93)$$

The Eddington tensor is a dimensionless tensor whose magnitude is always of order unity with $D_\nu \sim 1$, actually between $1/3$ and 1 , as explained in the previous sections. In our optically thick limit, since the radiation is close to isotropic, we have quite accurately

$$\mathbb{D}_\nu \simeq \frac{1}{3} \mathbb{I} \quad (3.94)$$

We define the light crossing time of the system as

$$t_{\text{cross}} = \frac{L}{c} \ll T = \frac{L}{V} \quad (3.95)$$

We see that the first term in the left-hand side is quite small, because $cT \gg L$ and $f_\nu \ll 1$. In this limit, we can safely ignore it in the momentum conservation equation, an approximation called the *stationary flux limit*. The term on the right-hand side, on the other hand, cannot be ignored. It is true that $f_\nu \ll 1$ but in the same time, we have $\lambda_\nu \ll L$, since we are in the optically thick regime. Following the Chapman-Enskog method, we just consider the two remaining terms have to be of the same order. This leads to the diffusion approximation, for which we have

$$\frac{c}{3} \nabla \cdot (\mathbb{I} E_\nu) = \frac{c}{3} \nabla E_\nu = -\alpha_\nu \mathbf{F}_\nu \text{ or } \boxed{\mathbf{F}_\nu = -\frac{c}{3\alpha_\nu} \nabla E_\nu} \quad (3.96)$$

The radiation energy conservation equation becomes

$$\frac{\partial E_\nu}{\partial t} = \nabla \cdot \left(\frac{c}{3\alpha_\nu} \nabla E_\nu \right) \quad (3.97)$$

which is nothing but the good old heat equation. We see another analogy with kinetic theory here, where energy transport is dominated by a diffusion process, and the diffusion coefficient $\nu = c\lambda_\nu$. The other source terms in the radiation energy equation vanished because we have $I_\nu \simeq B_\nu(T)$. Since we have

$$E_\nu \simeq \frac{4\pi}{c} B_\nu(T) \quad (3.98)$$

we can write

$$\nabla E_\nu \simeq \frac{4\pi}{c} \frac{\partial B_\nu}{\partial T} \nabla T \quad (3.99)$$

We can also integrate the previous equation over frequencies, and obtain

$$\frac{\partial E_{\text{rad}}}{\partial t} + \nabla \cdot \mathbf{F}_{\text{rad}} = 0 \quad (3.100)$$

where the frequency integrated radiative flux write

$$\mathbf{F}_{\text{rad}} = -\frac{c}{3\alpha_{\text{R}}} 4aT^3 \nabla T = -\kappa_{\text{rad}} \nabla T \quad (3.101)$$

where we introduce *the Rosseland mean* as

$$\boxed{\frac{1}{\alpha_{\text{R}}} = \left(\int_0^{+\infty} \frac{1}{\alpha_{\nu}} \frac{\partial B_{\nu}}{\partial T} d\nu \right) / \left(\int_0^{+\infty} \frac{\partial B_{\nu}}{\partial T} d\nu \right)} \quad (3.102)$$

Recall that we have for the black body radiation

$$\int_0^{+\infty} \frac{4\pi}{c} B_{\nu} d\nu = aT^4 \quad (3.103)$$

It is worth stressing that the radiative flux in the diffusion limit is very close to the heat flux we derived from kinetic theory, with a radiative heat coefficient

$$\kappa_{\text{rad}} = \frac{4acT^3}{3\alpha_{\text{R}}} \quad (3.104)$$

that can be added directly to the electron conduction coefficient, since both processes are proportional to the temperature gradient.

We can also compute the radiation force in the diffusion limit. It writes

$$\rho \mathbf{a}_{\text{rad}} = \int_0^{+\infty} \frac{\alpha_{\nu} \mathbf{F}_{\nu}}{c} d\nu = -\frac{1}{3} \int_0^{+\infty} \nabla E_{\nu} d\nu = -\frac{1}{3} \nabla E_{\text{rad}} = -\nabla P_{\text{rad}} \quad (3.105)$$

We conclude that in the diffusion limit, the radiation force is independent of the opacity, and is nothing but minus the radiation pressure gradient, completely analog to a fluid with $\gamma = 4/3$. Indeed, the momentum conservation equation for the fluid writes in the diffusion limit

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla (P + P_{\text{rad}}) = \rho \mathbf{a} \quad (3.106)$$

In the limit of infinite opacity, however, the radiation flux in the diffusion limit converges to zero, and the energy equation writes

$$\frac{\partial E_{\text{rad}}}{\partial t} = 0 \quad (3.107)$$

which is not what we expect from a $\gamma = 4/3$ fluid. This inconsistency comes from the fact that we have ignored so far relativistic corrections to the equations, due to the Lorentz transform from the comoving (or rest) frame to the laboratory (or observer) frame. We will discuss these interesting aspects in the next section.

3.6 Relativistic corrections for radiative transfer

Our derivation of the radiative transfer equations ignored completely so far relativistic effects of the combined fluid of interacting photons and particles. This is problematic in multiple ways. First, we have assumed that the emissivity coefficient j_{ν} was isotropic, because the underlying Maxwell-Boltzmann distribution of the particle velocities was isotropic. This is only true in

the frame comoving with the bulk flow. In the laboratory frame, the particle velocities are the sum of the random thermal velocities and the average, fluid velocities that points in one specific direction. If one defines the emissivity coefficient in the comoving frame as j_ν^0 , this is the truly isotropic one. But the emissivity coefficient in the laboratory frame j_ν should be modified by the Lorentz transformation bringing the emitted photon distribution to the lab frame.

The second problem arises when the radiation is close to LTE with matter. In this case, we have shown that the radiation specific intensity becomes isotropic, and equal to the source function, which, for thermal radiation, is equal to the black body distribution. Again, this is only true in the comoving frame. One must therefore define radiation moments in the comoving frame, as E_ν^0 and \mathbf{F}_ν^0 , which differ from their equivalent in the laboratory frame E_ν and \mathbf{F}_ν up to a Lorentz transform. Note that here only photons are relativistic particles, but the fluid is not, so we have $v/c \ll 1$. The corrections are likely to be very small. But as we have seen in the previous section, in the diffusion limit, the radiation flux is also very small. We therefore can't ignore these small relativistic corrections if we want to accurately describe the optically thick limit of radiative hydrodynamics. In the optically thin limit, relativistic effects can be ignored, at least for radiation transport. As we will see later, Doppler effects are important for line transfer and cannot be ignored, even in the optically thin limit.

3.6.1 Lorentz transform to first order in v/c

In order to change frame in the relativistic framework, we need to introduce the Lorentz transform. We however only keep terms up to first order in v/c , dropping all terms in $(v/c)^2$ and higher. This gives the following matrix

$$\mathbb{L} = \begin{bmatrix} 1 & -\frac{v_x}{c} & -\frac{v_y}{c} & -\frac{v_z}{c} \\ -\frac{v_x}{c} & 1 & 0 & 0 \\ -\frac{v_y}{c} & 0 & 1 & 0 \\ -\frac{v_z}{c} & 0 & 0 & 1 \end{bmatrix} \quad (3.108)$$

Note that this modified Lorentz transform is not unitary anymore but only to first order so that $|\det(\mathbb{L})| = 1 + \mathcal{O}((v/c)^2)$. A 4-vector coordinate in space-time in the comoving frame is related to the coordinate in the laboratory frame by

$$\mathbf{X}_0 = \mathbb{L}\mathbf{X} \text{ where } \mathbf{X}_0 = (ct_0, x_0, y_0, z_0)^T \text{ and } \mathbf{X} = (ct, x, y, z)^T \quad (3.109)$$

We obtain the classical Galilean transformation to the comoving frame, with however a first-order shift in the time coordinate

$$ct_0 = ct - \frac{\mathbf{v} \cdot \mathbf{x}}{c} \text{ and } \mathbf{x}_0 = \mathbf{x} - \mathbf{v}t \quad (3.110)$$

We can also apply the Lorentz transform to the photon 4-momentum

$$\mathbf{P}_0 = \mathbb{L}\mathbf{P} \text{ where } \mathbf{P}_0 = \left(\frac{h\nu_0}{c}, \frac{h\nu_0}{c} \mathbf{n}_0 \right)^T \text{ and } \mathbf{P} = \left(\frac{h\nu}{c}, \frac{h\nu}{c} \mathbf{n} \right)^T \quad (3.111)$$

where \mathbf{n}_0 and \mathbf{n} (resp. ν_0 and ν) are the unit vectors pointing in the direction of the radiation (resp. the photon frequency) in the comoving frame and in the laboratory frame. We obtain the Doppler effect as

$$\boxed{\nu_0 = \nu \left(1 - \frac{\mathbf{v} \cdot \mathbf{n}}{c} \right)} \quad (3.112)$$

and the angle aberration effect as

$$\boxed{\nu_0 \mathbf{n}_0 = \nu \left(\mathbf{n} - \frac{\mathbf{v}}{c} \right)} \quad (3.113)$$

From our discussion in the previous chapter on relativistic kinetic theory, we have shown that the phase-space fluid element $d^3x d^3p$ is a relativistic invariant, but d^3p alone is not. We have shown however that d^3p/E is a relativistic invariant. Using the form of d^3p derived at the beginning of this chapter, and using $E = h\nu$ for photons, we have

$$\frac{d^3p}{E} = \frac{p^2}{E} dp d\Omega = \frac{h^2 \nu}{c^3} d\nu d\Omega \quad (3.114)$$

we deduce that

$$\boxed{\nu_0 d\nu_0 d\Omega_0 = \nu d\nu d\Omega} \quad (3.115)$$

Using Liouville's theorem, we have also deduced that the distribution function f_ν is a relativistic invariant, but I_ν , the radiation specific intensity, is not. Using the relation we have derived between f_ν and I_ν , namely

$$I_\nu = \frac{h^4 \nu^3}{c^2} f_\nu \quad (3.116)$$

we deduce that I_ν/ν^3 is a relativistic invariant, so that

$$\boxed{\frac{I_\nu^0}{\nu_0^3} = \frac{I_\nu}{\nu^3}} \quad (3.117)$$

3.6.2 Moments equations in the laboratory frame

In order to account for the relativistic corrections for the emission and absorption coefficients, we need to consider the absorption along a light ray in the two frames. For simplicity, we assume that the velocity of the fluid is only along the z-axis with $\mathbf{v} = v\mathbf{e}_z$. We can write the two direction vectors \mathbf{n}_0 and \mathbf{n} using the spherical coordinate system along the z axis. We have

$$\begin{aligned} \nu_0 \sin \theta_0 \cos \phi_0 &= \nu \sin \theta \cos \phi \\ \nu_0 \sin \theta_0 \sin \phi_0 &= \nu \sin \theta \sin \phi \\ \nu_0 \cos \theta_0 &= \nu \left(\cos \theta - \frac{v}{c} \right) \end{aligned} \quad (3.118)$$

The optical depth along the light ray can then be computed in the laboratory frame as

$$d\tau = \alpha_\nu ds = \alpha_\nu \frac{d\ell}{\sin \theta} \quad (3.119)$$

where $d\ell = \sqrt{dx^2 + dy^2}$ is the projected length of the light ray ds in the $x - y$ plane. The key argument is as follows: the optical depth is a relativistic invariant, because it controls the number of photon absorbed by the slab along the light ray. We thus have $d\tau_0 = d\tau$. Because the Lorentz boost is only along the z direction, $d\ell$, perpendicular to the fluid motion, is not affected by the change of frame. We then have $d\ell_0 = d\ell$. On the other hand, the angle θ is affected by the angle aberration. Combining the two first equations on the spherical coordinates, we get

$$\nu_0 \sin \theta_0 = \nu \sin \theta \quad (3.120)$$

We finally get the relation between α_ν^0 and α_ν in the two frames as

$$\boxed{\nu_0 \alpha_\nu^0 = \nu \alpha_\nu} \quad (3.121)$$

The emissivity coefficient, j_ν can be treated similarly, by noticing that in the case of thermal radiation, one has $j_\nu = \alpha_\nu B_\nu$, where B_ν is the specific intensity of the black body radiation. It therefore also satisfies

$$\frac{B_\nu^0}{\nu_0^3} = \frac{B_\nu}{\nu^3} \quad (3.122)$$

Using the relation for α_ν , we deduce that

$$\boxed{\frac{j_\nu^0}{\nu_0^2} = \frac{j_\nu}{\nu^2}} \quad (3.123)$$

We can write the radiative transfer equation in the laboratory frame as usual

$$\frac{1}{c} \frac{\partial}{\partial t} I_\nu + \mathbf{n} \cdot \nabla I_\nu = j_\nu - \alpha_\nu I_\nu \quad (3.124)$$

We then integrate this equation both in angle and in frequency, taking as usual the first and second moments. For the first moment, we get easily

$$\frac{\partial}{\partial t} E_{\text{rad}} + \nabla \cdot \mathbf{F}_{\text{rad}} = \int_0^{+\infty} \int_{4\pi} j_\nu d\nu d\Omega - \int_0^{+\infty} \int_{4\pi} \alpha_\nu I_\nu d\nu d\Omega \quad (3.125)$$

We need to express the two terms on the right-hand side as a function of the emissivity and absorption coefficients in the comoving frame, the truly isotropic ones. The first term was defined as the cooling function. It can be written as

$$C_{\text{rad}} = \int_0^{+\infty} \int_{4\pi} j_\nu d\nu d\Omega = \int_0^{+\infty} \int_{4\pi} \frac{j_\nu}{\nu^2} \nu (\nu d\nu d\Omega) = \int_0^{+\infty} \int_{4\pi} \frac{j_\nu^0}{\nu_0^2} \nu (\nu_0 d\nu_0 d\Omega_0) \quad (3.126)$$

Using both the Doppler effect and the angle aberration, we obtain to leading order in v/c

$$C_{\text{rad}} \simeq \int_0^{+\infty} \int_{4\pi} j_\nu^0 \left(1 + \frac{\mathbf{v}}{c} \cdot \mathbf{n}_0\right) d\nu_0 d\Omega_0 = C_{\text{rad}}^0 = \alpha_{\text{P}}^0 c a T_{\text{gas}}^4 \quad (3.127)$$

where we assume thermal radiation for the emissivity coefficient in the comoving frame. The Doppler contribution integrates to zero because j_ν^0 is isotropic in the comoving frame. The second term on the right-hand side of the radiative energy equation was defined as the heating function. It can be expressed as a function of the absorption coefficient in the comoving frame using

$$\mathcal{H}_{\text{rad}} = \int_0^{+\infty} \int_{4\pi} \alpha_\nu I_\nu d\nu d\Omega = \int_0^{+\infty} \int_{4\pi} (\alpha_\nu \nu) \frac{1}{\nu} I_\nu d\nu d\Omega = \int_0^{+\infty} \int_{4\pi} \alpha_\nu^0 \frac{\nu_0}{\nu} I_\nu d\nu d\Omega \quad (3.128)$$

Using again the formula for the Doppler effect, we obtain

$$\mathcal{H}_{\text{rad}} = \int_0^{+\infty} \int_{4\pi} \alpha_\nu^0 \left(1 - \frac{\mathbf{v}}{c} \cdot \mathbf{n}\right) I_\nu d\nu d\Omega = \alpha_{\text{E}}^0 c E_{\text{rad}} - \alpha_{\text{F}}^0 \frac{\mathbf{v}}{c} \cdot \mathbf{F}_{\text{rad}} \quad (3.129)$$

We finally obtain the radiation energy equation in the laboratory frame as

$$\boxed{\frac{\partial}{\partial t} E_{\text{rad}} + \nabla \cdot \mathbf{F}_{\text{rad}} = \alpha_{\text{P}}^0 c a T_{\text{gas}}^4 - \alpha_{\text{E}}^0 c E_{\text{rad}} + \alpha_{\text{F}}^0 \frac{\mathbf{v}}{c} \cdot \mathbf{F}_{\text{rad}}} \quad (3.130)$$

For the second moment of the radiation transfer equation, we obtain

$$\frac{1}{c} \frac{\partial}{\partial t} \mathbf{F}_{\text{rad}} + c \nabla \cdot \mathbb{P}_{\text{rad}} = \int_0^{+\infty} \int_{4\pi} j_\nu \mathbf{n} d\nu d\Omega - \int_0^{+\infty} \int_{4\pi} \alpha_\nu I_\nu \mathbf{n} d\nu d\Omega \quad (3.131)$$

The first term on the right-hand side can be computed using like before the angle aberration formula. We get to leading order in v/c (left to the reader as an exercise)

$$\int_0^{+\infty} \int_{4\pi} j_\nu \mathbf{n} d\nu d\Omega \simeq \alpha_{\text{P}}^0 a T_{\text{gas}}^4 \mathbf{v} \quad (3.132)$$

We conclude that there is in fact a non-zero contribution of the emissivity coefficient to the radiation flux equation, entirely due to the Lorentz transform. The second term on the right-hand side is simpler to handle because we want to keep the radiation moments in the laboratory frame. We find

$$\int_0^{+\infty} \int_{4\pi} \alpha_\nu I_\nu \mathbf{n} d\nu d\Omega = \int_0^{+\infty} \int_{4\pi} \alpha_\nu^0 \left(1 - \frac{\mathbf{v} \cdot \mathbf{n}}{c}\right) I_\nu \mathbf{n} d\nu d\Omega \quad (3.133)$$

which can be simplified using the radiation flux and the radiation pressure tensor as

$$\int_0^{+\infty} \int_{4\pi} \alpha_\nu I_\nu \mathbf{n} d\nu d\Omega = \alpha_{\mathbf{F}}^0 \mathbf{F}_{\text{rad}} - \alpha_{\mathbf{E}}^0 \mathbb{P}_{\text{rad}} \mathbf{v} \quad (3.134)$$

We finally obtain for the radiation momentum equation in the laboratory frame

$$\boxed{\frac{1}{c} \frac{\partial}{\partial t} \mathbf{F}_{\text{rad}} + c \nabla \cdot \mathbb{P}_{\text{rad}} = \alpha_{\mathbf{P}}^0 a T_{\text{gas}}^4 \mathbf{v} - \alpha_{\mathbf{F}}^0 \mathbf{F}_{\text{rad}} + \alpha_{\mathbf{E}}^0 \mathbb{P}_{\text{rad}} \mathbf{v}} \quad (3.135)$$

We can check that if $F_{\text{rad}} \simeq c E_{\text{rad}}$, then the new correction terms are vanishingly small, justifying why we neglect them in the optically thin regime. In the optically thick regime, however, they are required.

3.6.3 Moments equations in the comoving frame

In order to simplify the discussion, we will now consider that we have a *grey material*, meaning that the opacity does not depend on frequency, so that all our different definitions (Rosseland, Planck, flux and energy means) for the frequency integrated opacities are equivalent. We can then simplify the previous two equations as

$$\frac{\partial}{\partial t} E_{\text{rad}} + \nabla \cdot \mathbf{F}_{\text{rad}} = \alpha_0 c (a T_{\text{gas}}^4 - E_{\text{rad}}) + \alpha_0 \mathbf{F}_{\text{rad}} \cdot \frac{\mathbf{v}}{c} \quad (3.136)$$

$$\frac{1}{c} \frac{\partial}{\partial t} \mathbf{F}_{\text{rad}} + c \nabla \cdot \mathbb{P}_{\text{rad}} = \alpha_0 a T_{\text{gas}}^4 \mathbf{v} - \alpha_0 \mathbf{F}_{\text{rad}} + \alpha_0 \mathbb{P}_{\text{rad}} \mathbf{v} \quad (3.137)$$

If one wants to properly converge to the diffusion limit, it is mandatory to define also the different radiation moments in the comoving frame. We thus need to transform also E_{rad} and \mathbf{F}_{rad} from the laboratory frame to the comoving frame. To perform this operation, we will use the relativistic invariance of d^3p/E and I_ν/ν^3 and write

$$E_{\text{rad}}^0 = \int_0^{+\infty} \int_{4\pi} \frac{I_\nu^0}{c} d\nu_0 d\Omega_0 = \int_0^{+\infty} \int_{4\pi} \frac{\nu_0^3}{\nu^3} \frac{I_\nu}{c} \frac{\nu}{\nu_0} d\nu d\Omega \quad (3.138)$$

Taylor expanding the Doppler term to first order in v/c , we finally get

$$E_{\text{rad}}^0 \simeq \int_0^{+\infty} \int_{4\pi} \frac{I_\nu}{c} \left(1 - 2 \frac{\mathbf{v} \cdot \mathbf{n}}{c}\right) d\nu d\Omega \quad (3.139)$$

We can perform the angular and frequency integration and obtain

$$\boxed{E_{\text{rad}}^0 = E_{\text{rad}} - \frac{2}{c} \mathbf{F}_{\text{rad}} \cdot \frac{\mathbf{v}}{c}} \quad (3.140)$$

Similarly, we can write the radiation flux in the comoving frame as

$$\mathbf{F}_{\text{rad}}^0 = \int_0^{+\infty} \int_{4\pi} I_\nu^0 \mathbf{n}_0 d\nu_0 d\Omega_0 = \int_0^{+\infty} \int_{4\pi} \frac{\nu_0^3}{\nu^3} I_\nu \frac{\nu^2}{\nu_0^2} \left(\mathbf{n} - \frac{\mathbf{v}}{c}\right) d\nu d\Omega \quad (3.141)$$

and obtain

$$\mathbf{F}_{\text{rad}}^0 = \int_0^{+\infty} \int_{4\pi} I_\nu \left(1 - \frac{\mathbf{v} \cdot \mathbf{n}}{c}\right) \left(\mathbf{n} - \frac{\mathbf{v}}{c}\right) d\nu d\Omega \quad (3.142)$$

Performing the angular and frequency integration and neglecting a quadratic term in v/c , we finally get

$$\boxed{\mathbf{F}_{\text{rad}}^0 = \mathbf{F}_{\text{rad}} - E_{\text{rad}}\mathbf{v} - \mathbb{P}_{\text{rad}}\mathbf{v}} \quad (3.143)$$

Injecting these two equations into the frequency integrated radiation energy and momentum conservation equations and dropping all quadratic terms in v/c leads to

$$\frac{\partial}{\partial t} E_{\text{rad}} + \nabla \cdot \mathbf{F}_{\text{rad}} \simeq \alpha_0 c (aT_{\text{gas}}^4 - E_{\text{rad}}^0) - \alpha_0 \mathbf{F}_{\text{rad}}^0 \cdot \frac{\mathbf{v}}{c} \quad (3.144)$$

$$\frac{1}{c} \frac{\partial}{\partial t} \mathbf{F}_{\text{rad}} + c \nabla \cdot \mathbb{P}_{\text{rad}} \simeq \alpha_0 (aT_{\text{gas}}^4 - E_{\text{rad}}^0) \mathbf{v} - \alpha_0 \mathbf{F}_{\text{rad}}^0 \quad (3.145)$$

where now we have only comoving variables in the right-hand side of these equations. We see now that the optically thick limit corresponds to $E_{\text{rad}}^0 \simeq aT_{\text{gas}}^4$ and $\mathbf{F}_{\text{rad}}^0 \simeq 0$ both expressed in the comoving frame.

We need to derive now the diffusion limit in term of the comoving radiation energy and flux, writing

$$E_{\text{rad}}^0 = aT_{\text{gas}}^4 (1 + e) \quad \text{and} \quad \mathbf{F}_{\text{rad}}^0 = caT_{\text{gas}}^4 (0 + \mathbf{f}) \quad (3.146)$$

where the reduced energy and the reduced flux are both small, with $|e| \ll 1$ and $|f| \ll 1$ but, very importantly, both small deviations are of the same order, with $|e| \sim |f|$. We perform the same order of magnitude estimate as before, on our new radiation momentum conservation equation, with typical fluid length scale L and time scale $T = L/V$,

$$\left[f \frac{caT_{\text{gas}}^4}{cT} \right] + \left[D_{\text{rad}} \frac{caT_{\text{gas}}^4}{L} \right] = - \left[e \frac{VaT_{\text{gas}}^4}{\lambda_0} \right] - \left[f \frac{caT_{\text{gas}}^4}{\lambda_0} \right] \quad (3.147)$$

The first term in the left-hand side can be dropped because we have $\mathbb{D}_{\text{rad}} = 1/3\mathbb{I}$ and $t_{\text{cross}} \ll T$ as before. The first term in the right-hand side can also be ignored, because $V \ll c$. In this Chapman-Enskog approach, we obtain for the diffusion limit the two relations

$$\mathbf{F}_{\text{rad}}^0 \simeq -\frac{c}{3\alpha_0} \nabla E_{\text{rad}}^0 \quad \text{and} \quad E_{\text{rad}}^0 \simeq aT_{\text{gas}}^4 \quad (3.148)$$

exactly as before, but now valid in the comoving frame. In the diffusion limit, we can also write the transformation between the laboratory frame and the comoving frame as

$$E_{\text{rad}}^0 \simeq E_{\text{rad}} \quad \text{and} \quad \mathbf{F}_{\text{rad}}^0 \simeq \mathbf{F}_{\text{rad}} - \frac{4}{3} E_{\text{rad}}^0 \mathbf{v} \quad (3.149)$$

We now inject these relations to obtain the radiation energy equation fully formulated in the comoving frame and in the diffusion limit as

$$\frac{\partial}{\partial t} E_{\text{rad}}^0 + \nabla \cdot \mathbf{F}_{\text{rad}}^0 + \nabla \cdot \left(\frac{4}{3} E_{\text{rad}}^0 \mathbf{v} \right) = \alpha_0 c (aT_{\text{gas}}^4 - E_{\text{rad}}^0) + \frac{1}{3} \mathbf{v} \cdot \nabla E_{\text{rad}}^0 \quad (3.150)$$

which can be re-arranged as

$$\boxed{\frac{\partial}{\partial t} E_{\text{rad}}^0 + \nabla \cdot (E_{\text{rad}}^0 \mathbf{v}) + \frac{1}{3} E_{\text{rad}}^0 \nabla \cdot \mathbf{v} = \nabla \cdot \left(\frac{c}{3\alpha_0} E_{\text{rad}}^0 \right) + \alpha_0 c (aT_{\text{gas}}^4 - E_{\text{rad}}^0)} \quad (3.151)$$

Note that the left-hand side of the radiation energy equation is strictly equivalent to the internal energy equation of a $\gamma = 4/3$ fluid. Relativistic corrections have thus repaired the inconsistency we have detected in the previous section. We can now study the infinite opacity limit $\alpha_0 \rightarrow +\infty$. We have in this case $E_{\text{rad}}^0 = aT_{\text{gas}}^4$ and $\mathbf{F}_{\text{rad}}^0 = 0$. We also have $T_{\text{rad}} = T_{\text{gas}} = T$. Combining the matter internal energy equation with the radiation energy equation, we can obtain the full LTE radiation hydrodynamics equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{v} = 0 \quad (3.152)$$

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) + \nabla P_{\text{tot}} = \rho \mathbf{a} \quad (3.153)$$

$$\frac{\partial E_{\text{tot}}}{\partial t} + \nabla \cdot (E_{\text{tot}} + P_{\text{tot}}) \mathbf{v} = \rho \mathbf{a} \cdot \mathbf{v} \quad (3.154)$$

where the total pressure is defined as

$$P_{\text{tot}} = P_{\text{gas}} + P_{\text{rad}} = \frac{\rho k_{\text{B}} T}{m} + \frac{1}{3} a T^4 \quad (3.155)$$

and the total energy is defined as

$$E_{\text{tot}} = \frac{1}{2} \rho v^2 + \frac{3}{2} \frac{\rho k_{\text{B}} T}{m} + a T^4 \quad (3.156)$$

We see that the equation of state for the coupled radiation + matter fluid is highly non-trivial. If the gas pressure dominates, we have $\gamma \simeq 5/3$, but if the radiation pressure dominates, the fluid behaves as if $\gamma \simeq 4/3$. We see that the transition occurs at a critical temperature given by

$$T_{\text{crit}} = \left(\frac{3\rho k_{\text{B}}}{am} \right)^{1/3} \simeq 3.2 \times 10^7 \left(\frac{\rho}{1 \text{ g/cc}} \right)^{1/3} \text{ K} \quad (3.157)$$

The centre of the Sun has a density of 150 g/cc and a temperature of 1.6×10^7 K, so that the gas pressure dominates over the radiation pressure. In higher mass stars, on the other hand, the temperature is much higher and the radiation pressure dominates. It is quite interesting to realise that the full LTE gas + radiation fluid follows the original Euler equations, so that most of the conclusion of the previous chapter on astrophysical fluid dynamics apply.

3.7 Matter and radiation interaction

We now switch to the second part of the Chapter on radiation. The goal is to describe the main emission and absorption processes encountered in astrophysics. We will use extensively the wave-particle duality of radiation. While in the first part, we have adopted mainly the point of view of photons and light rays, we will now focus more on the point of view of electromagnetic waves. At a few rare occasions, we will also adopt a more quantum mechanics approach with discrete particles and/or energy levels. The emission and absorption processes in astrophysics can be decomposed into three main groups: bound-bound radiation with atomic and molecular lines, bound-free radiation due to atomic ionisation processes, and finally free-free radiation due to free electrons interacting with radiation or with ions. We will describe all these processes in details, obtaining in each case detailed equations describing the associated phenomenon.

3.8 Larmor formula

The most important ingredient we need in this course is the radiation emitted by an accelerated charged particle, in most cases an electron. Note that this electron can be bound to an atom or freely moving in the continuum. In order to compute the emitted radiation from moving charges, we need to rely on the theory of electromagnetism and the use of Maxwell's equations and retarded potentials. In this section, we use a much simpler derivation due to Thomson.

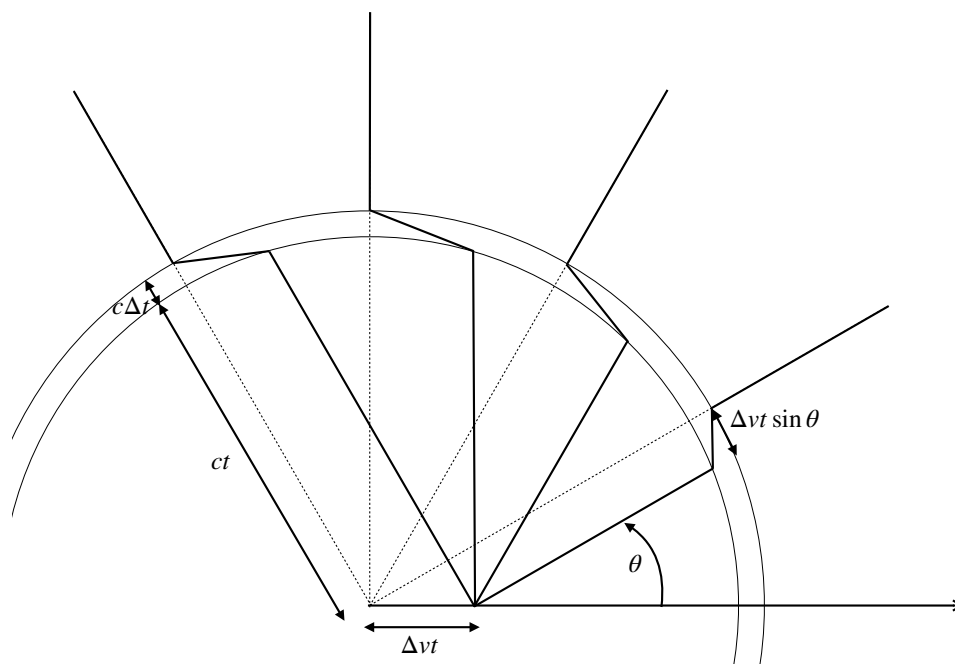


Figure 3.2: Field lines around an accelerated charge used in the derivation of Larmor formula.

We consider a point particle of charge $q = -e$, namely an electron. If this electron is at rest, it creates around itself a purely radial electric field given by Coulomb's law

$$E_r = \frac{q}{r^2} \quad (3.158)$$

We now give a kick *in the x-direction* to this particle, so that it acquires a small velocity $v_x = \Delta v \ll c$ during a small time step Δt . The corresponding impulsive acceleration at time

$t = 0$ is therefore

$$a = \frac{\Delta v}{\Delta t} \quad (3.159)$$

Once the particle reaches its terminal velocity Δv , it develops again a Coulomb potential, but the information about the new particle position propagates at finite speed (the speed of light) along the field lines. We see in Figure 3.2 a schematic of these field lines. The information about the start of the accelerating pulse reach a given point along a field line at distance $r = ct$. The new configuration of the field line is in place at a distance $c\Delta t$ later. In order to connect properly the field line before and after the pulse, the electrical field has to adjust, so that a small transverse component is created, whose amplitude has to be, for geometrical reasons (see Fig. 3.2),

$$\frac{E_{\perp}}{E_r} = \frac{\Delta v t \sin \theta}{c \Delta t} \quad (3.160)$$

Injecting the value of E_r and replacing t by r/c , we get

$$E_{\perp} = \frac{q a \sin \theta}{r c^2} \quad (3.161)$$

A distant observer will see this transverse component as a time-dependant electric field, in other words, an electromagnetic wave. Electromagnetic waves are solution of Maxwell's equations in vacuum, with \mathbf{E} and \mathbf{B} evolving as two coupled vector fields with $E = B$. The energy flux carried by the electromagnetic wave is the Poynting flux, in units of $[\text{erg cm}^{-2} \text{ s}^{-1}]$

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \quad (3.162)$$

In the case of our accelerating charge, the Poynting vector is directed in the radial direction, and its norm is

$$S = \frac{c}{4\pi} E^2 = \frac{d^2 W}{dA dt} = \frac{1}{4\pi} \frac{q^2 a^2 \sin^2 \theta}{r^2 c^3} \quad (3.163)$$

We see that the radiation is not isotropic. The $\sin^2 \theta$ dependance is typical of dipolar radiation. The total power (in units of $[\text{erg s}^{-1}]$) emitted by the accelerating charge can be computed by integrating the flux through the surface of a distant sphere at $r = R$. Using spherical coordinates, we obtain

$$\frac{dW}{dt} = \frac{q^2 a^2}{4\pi c^3} \int_{4\pi} \frac{\sin^2 \theta}{R^2} R^2 d\Omega = \frac{q^2 a^2}{2c^3} \int_0^{\pi} \sin^3 \theta d\theta \quad (3.164)$$

where quite logically the radius dependence drops out. We used the surface element on the sphere $dS = r^2 d\Omega = r^2 \sin \theta d\theta d\phi$. Solving the simple integral over θ and replacing $q = -e$, we obtain the formula of Larmor for the total emitted power by the accelerating electron

$$\boxed{\frac{dW}{dt} = \frac{2}{3} \frac{e^2}{c^3} |\mathbf{a}(t)|^2} \quad (3.165)$$

We derived this famous formula for the simple case of an impulsive acceleration, but this formula is more general and can be used for any acceleration profile $\mathbf{a}(t)$.

3.8.1 Radiation spectrum of an electromagnetic wave

In order to define the radiation spectrum of an electromagnetic wave, we need to use Fourier analysis. The Fourier transform of the electric field is defined as

$$\hat{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E(t) \exp^{-i\omega t} dt \quad (3.166)$$

and the inverse Fourier transform is defined as

$$E(t) = \int_{-\infty}^{+\infty} \hat{E}(\omega) \exp^{i\omega t} d\omega \quad (3.167)$$

We now use (again) Parseval's theorem to define the total energy of the wave as

$$\int_{-\infty}^{+\infty} E(t)^2 dt = 2\pi \int_{-\infty}^{+\infty} |\hat{E}(\omega)|^2 d\omega \quad (3.168)$$

Note that we use a particular convention for the Fourier transform. This convention can differ from the one adopted in other text books. Since the electric field is a real signal, we have $|\hat{E}(-\omega)| = |\hat{E}(\omega)|$, so we have

$$\int_{-\infty}^{+\infty} E(t)^2 dt = 4\pi \int_0^{+\infty} |\hat{E}(\omega)|^2 d\omega \quad (3.169)$$

We now use the Poynting vector to obtain the energy flux carried by the wave $S(t) = \frac{c}{4\pi} E(t)^2$. Integrating this in time and using Parseval's theorem, we obtain the total energy per unit area as

$$\frac{dW}{dA} = c \int_0^{+\infty} |\hat{E}(\omega)|^2 d\omega \quad (3.170)$$

We see that the right-hand side represents the spectral energy decomposition of the electromagnetic waves, also called the radiation spectrum, in $[\text{erg cm}^{-2} \text{ Hz}^{-1}]$ with

$$\boxed{\frac{d^2W}{dAd\omega} = c |\hat{E}(\omega)|^2} \quad (3.171)$$

3.8.2 Emitted radiation spectrum

Using the relation between the electric field and the acceleration we derived earlier, we get for our accelerated electron

$$\frac{d^2W}{dAd\omega} = \frac{e^2 \sin^2 \theta}{r^2 c^3} |\hat{a}(\omega)|^2 \quad (3.172)$$

where we introduce the Fourier transform of the acceleration

$$\hat{a}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(t) \exp^{-i\omega t} dt \quad (3.173)$$

Integrating this flux spectrum over the sphere at $r = R$, we get

$$\frac{dW}{d\omega} = \frac{e^2}{c^3} |\hat{a}(\omega)|^2 \frac{1}{R^2} \int_{4\pi} \sin^2 \theta R^2 d\Omega = 2\pi \frac{e^2}{c^3} |\hat{a}(\omega)|^2 \int_0^\pi \sin^3 \theta d\theta \quad (3.174)$$

The emitted radiation spectrum in $[\text{erg Hz}^{-1}]$ can be finally written as

$$\boxed{\frac{dW}{d\omega} = \frac{8\pi e^2}{3 c^3} |\hat{a}(\omega)|^2} \quad (3.175)$$

This radiation spectrum is emitted by one particle being accelerated during one event, usually a collision between an electron and a proton. Another collision will give rise to another pulse of radiation, with probably a different dipole direction (the x-axis defined before) and a different spectrum. In order to obtain the total emissivity of the fluid, one need to multiply by the rate

of these events per unit volume \dot{n}_{coll} . Assuming that the direction of these events is isotropically distributed, like for a Maxwell-Boltzmann distribution of particles, we can obtain the final emissivity coefficient by multiplying by \dot{n}_{coll} as

$$4\pi j_\omega = \dot{n}_{\text{coll}} \frac{dW}{d\omega} \quad (3.176)$$

in units of $[\text{erg cm}^{-3} \text{ s}^{-1} \text{ Hz}^{-1}]$, which is exactly the definition of the emissivity coefficient we gave in the previous section, except it was noted j_ν with $\omega = 2\pi\nu$

3.9 Bremsstrahlung

We now have all the necessary tools to compute our first emission process, Bremsstrahlung, also called *free-free radiation*. It corresponds to Larmor radiation during collisions between free electrons and ions (mostly protons), interacting through the Coulomb interaction. Coulomb collisions have been studied in details in the first Chapter. Because ions are much more massive than electrons, the frame of the center of mass corresponds to the rest frame of the ion, while the relative velocity corresponds to the velocity of the electron. We assume that the ions have a charge $q = +Ze$, where $Z = 1$ for protons, while electrons have a charge $q = -e$.

We choose the x-axis aligned with the velocity of the incoming electron, while its y-coordinate at infinity is the impact parameter b . We use the small deflection angle approximation, so that the electron follows a straight line with $y \simeq b$ and $x \simeq vt$. The time coordinate is chosen so that $t = 0$ corresponds to the collision itself, where the radius between the electron and the ion is minimum with $r = b$, while before or after the collision the radius is $r^2 = x^2 + y^2 \simeq (vt)^2 + b^2$.

The acceleration that matters here is the acceleration in the y-direction. The acceleration in the x-direction changes sign at $t = 0$ and averages to zero. For the Coulomb force, it writes

$$m_e a(t) = -\frac{Ze^2}{r^3} y \simeq -\frac{Ze^2}{b^2} \left[1 + \left(\frac{t}{\tau} \right)^2 \right]^{-3/2} \quad \text{where } \tau = \frac{b}{v} \quad (3.177)$$

The total velocity kick in the y-direction is obtained by integrating the acceleration as

$$\Delta v = \int_{-\infty}^{+\infty} a(t) dt = -\frac{2Ze^2}{m_e b v} \quad (3.178)$$

We see that the acceleration has a typical bell shape, with a maximum at $t = 0$ and a thickness of $\tau = b/v$. The Fourier transform $\hat{a}(\omega)$ has also a typical bell shape with its maximum at $\omega = 0$. We have

$$\hat{a}(\omega) \simeq \frac{\Delta v}{2\pi} \quad \text{for } \omega \leq \frac{1}{\tau} \quad \text{and } \hat{a}(\omega) \simeq 0 \quad \text{for } \omega \geq \frac{1}{\tau} \quad (3.179)$$

Using Larmor formula, we compute the radiation energy spectrum of this single collision

$$\frac{dW}{d\omega} = \frac{8\pi}{3} \frac{e^2}{c^3} |\hat{a}(\omega)|^2 = \frac{2}{3\pi} \frac{e^2}{c^3} \Delta v^2 = \frac{8}{3\pi} \frac{Z^2 e^6}{c^3 m_e^2 b^2 v^2} \quad \text{for } \omega \leq \frac{v}{b} \quad (3.180)$$

Using the collision cylinder, we can compute the number of collisions between incoming electrons with impact parameter between b and $b + db$ and velocity at infinity v and $v + dv$ as

$$\frac{dn_{\text{coll}}}{dt} = n_i (n_e(v) dv) (2\pi b db) v \quad (3.181)$$

where $n_e(v)$ is given by the Maxwell-Boltzmann distribution. In order to compute the total emissivity coefficient, we need to integrate the single collision energy spectrum over all possible impact parameters and all possible incoming electron velocities.

$$4\pi j_\omega = \int_0^{+\infty} n_i n_e(v) v dv \int_0^{+\infty} 2\pi b \frac{dW}{d\omega} db \quad (3.182)$$

We first perform the integration with respect to the impact parameter. The functional form of the integrand is db/b , which integrates into a logarithm, and therefore, as always, diverges for both small and large impact parameters. Since the spectrum is approximately zero for $b \geq v/\omega$, we use as upper bound of the integral $b_{\max} = v/\omega$. For the lower bound, we use the usual value $b_{\min} = b_{90}$, the impact parameter corresponding to a 90-degree deflection angle. We then obtain

$$4\pi j_\omega = \frac{16}{3} \frac{Z^2 e^6}{c^3 m_e^2} \int_0^{+\infty} n_i n_e(v) \ln \left(\frac{b_{\max}}{b_{\min}} \right) \frac{dv}{v} \quad (3.183)$$

The logarithm is absorbed in a slowly varying quantity called the *free-free Gaunt factor* and defined by

$$g_{\text{ff}}(v, \omega) = \frac{\sqrt{3}}{\pi} \ln \left(\frac{b_{\max}}{b_{\min}} \right) \quad (3.184)$$

The factor $\sqrt{3}/\pi$ was introduced in the Gaunt factor to account for the exact form of the Fourier transform of the acceleration. The Gaunt factor is a slowly varying function of v and ω . The emitted spectrum from a single collision is therefore almost flat. This cannot be true for arbitrary high photon frequencies. Indeed, radiation during the collision will be emitted into discrete photons. Since the maximum available energy in the collision is the kinetic energy of the electron, we conclude that the maximum photon energy that can be emitted corresponds to one single photon with energy $h\nu = \frac{1}{2} m_e v^2$. This semi-classical argument allows us to derive the spectral shape of the emitted radiation. The electron distribution function derives from Maxwell-Boltzmann with

$$n_e(v) = \frac{n_e}{\left(2\pi \frac{k_B T}{m_e}\right)^{3/2}} \exp^{-\frac{m_e v^2}{2k_B T}} 4\pi v^2 \quad (3.185)$$

We can integrate our almost flat radiation spectrum in velocity space, only including electrons for which the kinetic energy is larger than the radiation energy

$$\frac{1}{2} m_e v^2 \geq h\nu \quad (3.186)$$

This sets the lower bound of the integral, and we obtain

$$4\pi j_\omega = \frac{32\pi}{3\sqrt{3}} \frac{Z^2 e^6}{c^3 m_e^2} \frac{n_i n_e}{\left(2\pi \frac{k_B T}{m_e}\right)^{1/2}} \exp^{-\frac{h\nu}{k_B T}} \overline{g_{\text{ff}}}(\nu) \quad (3.187)$$

The *average Gaunt factor* $\overline{g_{\text{ff}}}(\nu)$ corresponds to the previously defined Gaunt factor but averaged over the Maxwellian distribution. Finally, using $j_\nu = 2\pi j_\omega$, we get the emissivity coefficient of Bremsstrahlung

$$\boxed{4\pi j_\nu = \frac{32\pi}{3\sqrt{3}} \frac{Z^2 e^6}{c^3 m_e^2} n_i n_e \left(\frac{2\pi m_e}{k_B T} \right)^{1/2} \exp^{-\frac{h\nu}{k_B T}} \overline{g_{\text{ff}}}(\nu)} \quad (3.188)$$

We can integrate over the frequencies to compute the corresponding cooling function,

$$\mathcal{C}_{\text{ff}} = \int_0^{+\infty} 4\pi j_\nu d\nu = \frac{32\pi}{3\sqrt{3}} \frac{Z^2 e^6}{h c^3 m_e^2} n_i n_e (2\pi m_e k_B T)^{1/2} \overline{\overline{g_{\text{ff}}}} \quad (3.189)$$

Numerically, we obtain the final and famous result

$$\boxed{C_{\text{ff}} = 1.42 \times 10^{-27} \overline{g_{\text{ff}}} Z^2 n_i n_e \sqrt{T}} \quad (3.190)$$

The frequency average of the average Gaunt factor encodes the slow variations of the Gaunt factor $g_{\text{ff}}(v, \omega)$. This Gaunt factor is also used to include possible quantum corrections to our simple semi-classical treatment. It is usually close to unity and traditionally set to be $\overline{g_{\text{ff}}} \simeq 1.2$

In order to obtain the corresponding absorption coefficient, we invoke Kirchhoff's relation for thermal radiation

$$\alpha_\nu = \frac{j_\nu}{B_\nu(T)} \quad (3.191)$$

where $B_\nu(T)$ is the black body specific intensity. Note that the Bremsstrahlung emission has a characteristic spectral shape, with an exponential cut-off for $h\nu \gg k_B T$ and a flat spectrum for $h\nu \ll k_B T$. The cooling function has a characteristic \sqrt{T} temperature dependence. Bremsstrahlung is the only remaining cooling process for hot ionised gas with $T > 10^7$ K. At high photon frequencies, because of the exponential cut-off, the only remaining absorption process is Thomson scattering.

3.10 Thomson scattering

Thomson scattering describes the interaction between a planar electromagnetic wave with a free electron in the plasma. It can be described fully classically, as long as the energy of the incoming photons satisfies $h\nu \ll m_e c^2$. In short, the incoming electromagnetic wave couples to the electron that starts to oscillate. The resulting electron acceleration is transformed into Larmor radiation, emitted in a dipole pattern. The corresponding energy is removed from the incident wave, leading to the absorption of the incident radiation, and, simultaneously, to the emission of radiation in the same frequency but in a different direction. This is why this process is called *scattering*. Before we describe in details the interaction between the planar wave and the free electron, we first need to introduce planar electromagnetic waves.

3.10.1 Planar electromagnetic waves

A planar wave propagating in the direction of a unit vector \mathbf{n}_0 is described by the following electrical field

$$\mathbf{E}(x, t) = \mathbf{E}_0 \exp^{i(k_0 x - \omega_0 t)} = E_0 \exp^{i(k_0 x - \omega_0 t)} \mathbf{e}_0 \quad (3.192)$$

where the x-axis is chosen aligned with \mathbf{n}_0 . The unit vector \mathbf{e}_0 indicates the direction of the *polarisation* of the wave, perpendicular to the x-axis. The wave is monochromatic with frequency ω_0 and satisfies the dispersion relation of waves propagating in vacuum $\omega_0 = ck_0$. The magnetic field follows a similar form with $\mathbf{B} = B\mathbf{b}_0$ where $E = B$ and \mathbf{b}_0 is perpendicular to \mathbf{e}_0 and the x-axis. The energy flux carried by the wave is given by the Poynting flux

$$\mathbf{S}(x, t) = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{c}{4\pi} E B \mathbf{e}_0 \times \mathbf{b}_0 = \frac{c}{4\pi} E B \mathbf{e}_x \quad (3.193)$$

We see that the Poynting flux is aligned with the x-axis and its norm can be written as

$$S(x, t) = \frac{c}{4\pi} E^2(x, t) \quad (3.194)$$

It is a space- and time-dependent quantity, that oscillates at twice the frequency of the electric field. Planar waves are analogues to the light-rays we used in the previous chapter. The radiation specific intensity of a light ray propagating in vacuum is however constant, and not oscillatory.

The usual way of solving this conundrum is to define the intensity of the planar wave as the average Poynting flux over one period $T_0 = 2\pi/\omega_0$ and at some fixed location x . Adopting without loss of generality $x = 0$, we have

$$\mathbf{E}(x = 0, t) = \mathbf{E}_0 \cos(\omega_0 t) \quad (3.195)$$

The average Poynting flux is a constant along the planar wave and its value is

$$S_0 = \frac{1}{T_0} \int_0^{T_0} \frac{c}{4\pi} E_0^2 \cos^2(\omega_0 t) dt = \frac{c}{8\pi} E_0^2 \quad (3.196)$$

We can now relate the specific radiation intensity along the light ray to the planar wave energy flux as

$$I_\omega = S_0 \delta(\omega - \omega_0) \delta(\mathbf{n} - \mathbf{n}_0) \quad (3.197)$$

We see that although S_0 has units of $[\text{erg cm}^{-2}\text{s}^{-1}]$, I_ω has units of $[\text{erg cm}^{-2}\text{s}^{-1}\text{Hz}^{-1}\text{str}^{-1}]$.

3.10.2 Thomson cross section

We now consider the Lorentz force due to the electromagnetic wave acting on the free electron. The equation of motion for the electron writes

$$m_e \ddot{\mathbf{x}} = -e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \simeq -e \mathbf{E} \text{ so that } \ddot{\mathbf{x}} = -\frac{e}{m_e} E_0 \cos(\omega_0 t) \mathbf{e}_0 \quad (3.198)$$

because the electron velocity is non relativistic $v \ll c$ and $B = E$. From the previous sections, we know that this acceleration will be transformed into Larmor radiation, where the power emitted per solid angle is

$$\frac{dW}{dt d\Omega}(t) = \frac{e^2 \ddot{\mathbf{x}}^2(t)}{4\pi c^3} \sin^2 \theta \quad (3.199)$$

The new electromagnetic wave has the same frequency of the incoming one, it is therefore also monochromatic, and the radiation intensity is defined using the average Poynting flux over one period. We therefore obtain

$$\frac{dW_0}{dt d\Omega} = \frac{e^4 E_0^2}{8\pi c^3 m_e^2} \sin^2 \theta \quad (3.200)$$

The angle θ is defined with respect to the polarisation of the incoming wave \mathbf{e}_0 . Integrating over solid angle, we get the power emitted by the oscillating electron in all direction averaged over one period

$$\frac{dW_0}{dt} = \frac{e^4 E_0^2}{3c^3 m_e^2} \quad (3.201)$$

In order to get the corresponding emissivity coefficient, we have to multiply by the number density of electrons, and by the Dirac delta function to account for the monochromaticity, which gives

$$4\pi j_\omega = n_e \frac{dW_0}{dt} \delta(\omega - \omega_0) \quad (3.202)$$

This energy emitted per unit time corresponds to the energy lost by the incoming wave. This allows us to compute the cross section as

$$\frac{dW_0}{dt} = \sigma_T \frac{c}{8\pi} E_0^2 \quad (3.203)$$

which leads to the Thomson cross-section

$$\boxed{\sigma_T = \frac{8\pi}{3} \frac{e^4}{c^4 m_e^2}} \quad (3.204)$$

The corresponding absorption coefficient is obtained using

$$\alpha_\nu = n_e \sigma_T \quad (3.205)$$

There is a simple way to obtain this result, using the *electron classical radius*, which can be defined by

$$\frac{e^2}{r_e} = m_e c^2 \quad (3.206)$$

which states that the electron electrostatic energy is equal to the energy associated to the electron rest mass. Although this has no particular physical meaning, Thomson scattering can be pictured as electrons of finite size blocking the light, like small dust grains, giving the cross section

$$\sigma_T \simeq \pi r_e^2 = \pi \frac{e^4}{c^4 m_e^2} \quad (3.207)$$

We are just missing the factor 8/3.

3.11 Atomic and molecular excitation levels

We now move to the description of what is called *bound-bound radiation*, namely absorption and emission processes associated to bound states of atoms and molecules. The atomic structure is usually described using quantum mechanics. This is not the goal of these lectures to give an accurate account of quantum processes. We will just review the main results of quantum physics theory, outlining a few well-known results using very rough derivations. We will use like in the previous section a semi-classical approach of the theory of radiation and matter interaction, leaving again the more rigorous derivations to quantum mechanics books.

3.11.1 Electronic, vibrational and rotational states

We consider bound states of the Hydrogen atom, with an electron orbiting around a proton. So far we have considered only free electrons and naked protons. What we know from quantum mechanics is that the electron orbital structure around the proton is quantized into a discrete set of states, called electronic states. The same is true for molecular processes, with vibrational and rotational states. We would like to estimate the energy of these electronic, vibrational and rotational states.

We can write the electron orbital energy as the sum of the potential energy and the kinetic energy.

$$E = \frac{p^2}{2m_e} - \frac{e^2}{r} \quad (3.208)$$

If the electron is on a bound orbit, it must have $E < 0$. We know from quantum mechanics that we cannot adopt a classical approach. The trick is to use Heisenberg uncertainty principle to derive an order of magnitude estimate of the electron momentum with $pr \simeq \hbar$. Injecting this relation for a marginally bound orbit with $E = 0$, we obtain the Bohr radius with

$$\frac{e^2}{r_B} = \frac{p_B^2}{2m_e} \text{ with } p_B = \frac{\hbar}{r_B} \quad (3.209)$$

This gives for the Bohr radius (note that we dropped the factor 2)

$$r_B = \frac{\hbar^2}{m_e e^2} \quad (3.210)$$

This is the typical radius for the electron orbit around the proton. The corresponding energy is (note that we keep now the factor 2)

$$E_{\text{elec}} = \frac{p_B^2}{2m_e} = \frac{\hbar^2}{2m_e r_B^2} = \frac{m_e e^4}{2\hbar^2} \quad (3.211)$$

The total energy for the bound electronic state for the Hydrogen atom is given by

$$E_{\text{elec},n} = -\frac{E_{\text{elec}}}{n^2} = -\frac{13.6 \text{ eV}}{n^2} \quad (3.212)$$

Using a similar approach, we can estimate the vibrational energy levels for molecules, more importantly the Hydrogen molecule. The idea is to consider the Hydrogen molecule as an harmonic oscillator, like a spring, whose equation of motion is given by $m_p \ddot{x} = -k_e x$ where the spring constant k_e corresponds to the potential of the molecular bound, estimated here to be roughly equal to the electrostatic potential e^2/r_B , and the typical spring length is given by the Bohr radius r_B . The harmonic oscillator energy in quantum mechanics is given by

$$E_{\text{vib}} = \hbar\omega \text{ where } \omega = \sqrt{\frac{k_e}{m_p}} \quad (3.213)$$

The spring constant can be determined requiring that the spring potential energy is equal to the electrostatic energy

$$\frac{1}{2}k_e r_B^2 = E_B = \frac{\hbar^2}{2m_e r_B^2} \text{ which gives } k_e = \frac{\hbar^2}{m_e r_B^4} \text{ and } \omega = \frac{\hbar}{\sqrt{m_e m_p} r_B^2} \quad (3.214)$$

We finally find the typical energy for vibrational levels as

$$E_{\text{vib}} = \frac{\hbar^2}{\sqrt{m_e m_p} r_B^2} = \sqrt{\frac{m_e}{m_p}} E_{\text{elec}} \quad (3.215)$$

and the corresponding discrete energy levels are given

$$E_{\text{vib},n} = E_{\text{vib}} \left(n + \frac{1}{2} \right) = 0.31 \text{ eV} \left(n + \frac{1}{2} \right) \quad (3.216)$$

Finally, the third category of bound states are molecular rotational energy levels, corresponding to the balance between the centrifugal force and the molecular bound. The kinetic energy associated to the rotation is now $K = \frac{1}{2}I\Omega^2 = \frac{1}{2}L^2/I$ where $I = m_p r_B^2$ is the moment of inertia of the molecule and $L = I\Omega$ its angular momentum. We can estimate roughly the typical angular momentum using the Heisenberg uncertainty principle $L \simeq \hbar$, which gives the typical energy for rotational levels

$$E_{\text{rot}} = \frac{L^2}{2m_p r_B^2} = \frac{\hbar^2}{2m_p r_B^2} = \frac{m_e}{m_p} E_{\text{elec}} \quad (3.217)$$

and the corresponding discrete energy levels

$$E_{\text{rot},n} = E_{\text{rot}} J(J+1) = 7.4 \times 10^{-4} \text{ eV} J(J+1) \quad (3.218)$$

We can compute the corresponding wavelength of the radiation associated to each process. Using $E = h\nu$, we obtain $\lambda = 912\text{\AA}$ for $E_{\text{elec}} = 13.6 \text{ eV}$, which corresponds to ultraviolet radiation. Vibrational levels have $\lambda \simeq 4 \mu\text{m}$ for $E_{\text{vib}} \simeq 0.3 \text{ eV}$ and emits infrared radiation, while rotational levels have $\lambda \simeq 0.2\text{mm}$ for $E_{\text{rot}} \simeq 10^{-3} \text{ eV}$ and are detected in the sub-mm range. Using $E = k_B T$ (a temperature of 11605 K corresponds to 1 eV), we see that atomic transitions are excited for temperature $T \simeq 10^5 \text{ K}$, while molecular vibrational levels are excited for $T \simeq 3000 \text{ K}$ and the lower rotational levels corresponds to $T \simeq 10 \text{ K}$.

3.11.2 Collisional excitation

The first and most simple process that drives the bound electron from a lower to a higher excited state are collisions. We have studied collisions quite extensively in the first Chapter. These collisions were elastic collisions, leading to a redistribution of the collision partners in momentum space, but without changing the nature of the involved particles. In this Chapter, we consider *inelastic collisions*, between say an electron and an atom in excited level i , leading to the excitation of the bound electron to a higher level j . In chemistry notations, this can be written as



The upper index usually refers to the charge of the particle, where the Hydrogen atom with the bound electron has zero net charge. The free electron, on the other hand, has a negative charge equal to $-e$, hence the upper index with a single negative sign. The double arrow in the centre mean that the reaction is reversible. One collision can promote the bound electron to the higher level j , but another later collision can demote the electron to the lower level i . In what follows,

for sake of simplicity, we will use a *two levels atom model*. The generalisation to a multiple levels atom is conceptually straightforward but computational more involving. We can now compute the creation rate of the Hydrogen atom in state $j = 1$ as

$$\frac{dn_1}{dt} = n_2 n_e \langle \sigma_{21} v \rangle - n_1 n_e \langle \sigma_{12} v \rangle \quad (3.220)$$

where the electron number density, and the number density of atoms in state 1 and 2 are computed as

$$n_e(\mathbf{x}, t) = \int_{\mathbb{R}^3} f_e d^3p \text{ and } n_1(\mathbf{x}, t) = \int_{\mathbb{R}^3} f_1 d^3p \text{ and } n_2(\mathbf{x}, t) = \int_{\mathbb{R}^3} f_2 d^3p \quad (3.221)$$

The two velocity-averaged cross sections are defined as the reaction cross section averaged over the distribution function of the relative velocity v , which is in most cases the distribution function of the electron. These collision rates are traditionally defined as

$$C_{21} = n_e \langle \sigma_{21} v \rangle \text{ and } C_{12} = n_e \langle \sigma_{12} v \rangle \quad (3.222)$$

so that the rate equation can now be written as

$$\boxed{\frac{dn_1}{dt} = C_{21}n_2 - C_{12}n_1 = -\frac{dn_2}{dt}} \quad (3.223)$$

Note that when the system is in chemical equilibrium, we have the relation

$$C_{21}n_2 = C_{12}n_1 \quad (3.224)$$

We will see in the following sections that this purely collisional description is only valid at high electron densities. At lower densities, we must also include radiative processes.

3.11.3 Level population at LTE

We now want to compute the distribution of the excited states of the Hydrogen atom at LTE. For this, we need first to define the energy of the Hydrogen atom occupying excitation level i as

$$E = \frac{1}{2}m_p v^2 + \frac{1}{2}m_e v^2 + E_{\text{elec},i} \quad (3.225)$$

where $E_{\text{elec},i}$ represents to total orbital energy of the bound electron, namely the sum of the orbital kinetic energy and the orbital electrostatic energy. In the previous equation, the first two terms represent the translational kinetic energy of the bound system, in which the velocity of the proton and the electron are the same. We can clearly neglect the contribution of the latter, because $m_p = 1833m_e$. We derived the LTE distribution function through reversible, elastic collisions imposing detailed balance in phase-space. We can do the same thing here, leading to distribution function for each species, with

$$f_e = \frac{2}{h^3} \exp^{(\mu_e - \frac{p^2}{2m_e})/k_B T} \quad (3.226)$$

for free electrons, and

$$f_i = \frac{g_i}{h^3} \exp^{(\mu_i - \frac{p^2}{2m_p} - E_i)/k_B T} \quad (3.227)$$

for the Hydrogen atoms in different excitation levels, where g_i is the maximum number of degenerate states of particle of type i that can occupy the same position in phase-space. Note

that we ignored here the Fermi suppression factor $1 - \mathcal{N}$: We deal here with a non-degenerate fluid. We use the same detailed balance argument, but now for the inelastic collisions between different species, $\ln f_e + \ln f_i = \ln f'_e + \ln f'_j$. Note that here detailed balance is performed for each individual degenerate state individually. Degenerate states have the same chemical potential, as long as they are indistinguishable. This translates into the condition

$$\boxed{\mu_e + \mu_i = \mu_e + \mu_j \text{ or equivalently } \mu_i = \mu_j} \quad (3.228)$$

because the total energy is conserved during the collision. Note that this additive property of the chemical potentials of reactants and products of a chemical reaction is quite general and of course true only at LTE. Integrating over momentum space, we get the total number density of atoms species i

$$n_i(\mathbf{x}, t) = \int_{\mathbb{R}^3} f_i(\mathbf{x}, \mathbf{p}, t) d^3p = g_i \exp^{-E_i/k_B T} \frac{\exp^{\mu_i/k_B T}}{h^3} \int_{\mathbb{R}^3} \exp^{-\frac{p^2}{2m_p}/k_B T} d^3p \quad (3.229)$$

The last integral over momentum space can be readily integrated (it is a Gaussian), and we obtain

$$n_i(\mathbf{x}, t) = g_i \exp^{-E_i/k_B T} \frac{\exp^{\mu_i/k_B T}}{h^3} (2\pi m_p k_B T)^{3/2} \quad (3.230)$$

Dividing the number densities of two different levels i and j , we see that the temperature and the chemical potential terms disappear (because $\mu_i = \mu_j$). We obtain the Boltzmann distribution of level populations at LTE

$$\boxed{\frac{n_i}{n_j} = \frac{g_i}{g_j} \exp^{-(E_i - E_j)/k_B T}} \quad (3.231)$$

We can also define the total number density of Hydrogen atoms, summing up over all possible excited states

$$n_{\text{tot}}(\mathbf{x}, t) = \sum_{i=1}^{+\infty} n_i(\mathbf{x}, t) = \left(\sum_{i=1}^{+\infty} g_i \exp^{-E_i/k_B T} \right) \frac{\exp^{\mu_1/k_B T}}{h^3} (2\pi m_p k_B T)^{3/2} \quad (3.232)$$

where we used the fact that $\mu_i = \mu_1$ for all levels. We can thus rewrite the level population at LTE as

$$\frac{n_i}{n_{\text{tot}}} = \frac{g_i \exp^{-E_i/k_B T}}{Z(T)} \text{ where } Z(T) = \sum_{i=1}^{+\infty} g_i \exp^{-E_i/k_B T} \quad (3.233)$$

The function $Z(T)$ is called the *partition function* of the level population. The first level $i = 1$, also called the ground state or the fundamental level has energy $E_1 = -E_{\text{elec}} = -13.6$ eV. For gas temperature $k_B T \ll E_{\text{elec}}$, we see that for all levels $i > 1$, the exponential term in the Boltzmann relation will be zero. In this regime, $n_{\text{tot}} \simeq n_1$ and the level population is said to be in the ground state. In the opposite regime, if $k_B T \gg E_{\text{elec}}$, then the exponential becomes one. The partition function now diverges. The Hydrogen atom has indeed multiple degenerate states for each electronic level with $g_i = i^2$, these degenerate states corresponding to angular momentum and spin degrees of freedom. In this limit, at high temperature, we get the inconsistent result that

$$Z(T) \simeq \sum_{i=1}^{+\infty} i^2 \rightarrow +\infty \quad (3.234)$$

We can solve this inconsistency with an analogy to the orbital model of the electron orbiting the proton. If we write that each energy level E_i corresponds to an orbital radius r_i , we find

$$E_i = -\frac{E_{\text{elec}}}{i^2} = -\frac{e^2}{r_i} \text{ we find } r_i = r_B i^2 \quad (3.235)$$

We now can compute the mean interparticle separation for the atoms and identify it as the last orbit the atom can have without having its bound electron being captured by another nearby atom.

$$r_{\max} = \frac{1}{n_{\text{tot}}^{1/3}} = r_{\text{B}} i_{\max}^2 \quad (3.236)$$

The partition function is then computed using only surviving bound states

$$Z(T) \simeq \sum_{i=1}^{i_{\max}} i^2 \quad \text{where } i_{\max} = \frac{1}{\sqrt{r_{\text{B}} n_{\text{tot}}^{1/3}}} \quad (3.237)$$

The partition function and the level population will converge properly, at the expense of having more and more free electrons ejected from the higher energy levels. This process is called *pressure ionisation*. Coming back to our simple two levels atom, we can compare the chemical equilibrium relation to the LTE level population

$$\frac{n_2}{n_1} = \frac{C_{12}}{C_{21}} \quad \text{versus} \quad \frac{n_2}{n_1} = \frac{g_2}{g_1} \exp^{-(E_2-E_1)/k_{\text{B}}T} \quad (3.238)$$

We know that at LTE, both relations are valid. Since C_{12} and C_{21} only depend on the microscopic cross section and on the electron Maxwell-Boltzmann distribution, we conclude that the collisional rates must satisfy this at all time (even in non-LTE conditions) the following relation

$$\boxed{\frac{C_{12}}{C_{21}} = \frac{g_2}{g_1} \exp^{-(E_2-E_1)/k_{\text{B}}T}} \quad (3.239)$$

In many astrophysical databases, only $\langle \sigma_{21} v \rangle$ is given, the collision rate from a higher to a lower level, in units of $[\text{cm}^3 \text{s}^{-1}]$. It usually depends quite weakly on the temperature. We can evaluate the collisional excitation cross section, using the hard sphere model with $\sigma_0 \simeq 10^{-15} \text{ cm}^2$ and requiring the incoming electron to have a kinetic energy at least equal to the excitation energy $\Delta E_{21} = E_2 - E_1$. This leads to the formula

$$C_{12} = n_e \langle \sigma_{12} v \rangle = \sigma_0 \int_{\Delta E_{21}}^{+\infty} v f_e(v) dv \simeq n_e \sigma_0 \left(\frac{k_{\text{B}}T}{m_e} \right)^{1/2} \exp^{-(E_2-E_1)/k_{\text{B}}T} \quad (3.240)$$

We can evaluate numerically the collision rate of this simple hard sphere model

$$\langle \sigma_{12} v \rangle \simeq 3.9 \times 10^{-10} T^{1/2} \exp^{-(E_2-E_1)/k_{\text{B}}T} \text{ cm}^3 \text{ s}^{-1} \quad (3.241)$$

The inverse reaction rate, from the lower to the higher level, can then be computed using the previous LTE-based relation. We obtain easily

$$C_{21} = n_e \frac{g_1}{g_2} \sigma_0 \left(\frac{k_{\text{B}}T}{m_e} \right)^{1/2} \quad (3.242)$$

where the exponential term nicely cancels out. The corresponding collision deexcitation rate writes

$$\langle \sigma_{21} v \rangle = 3.9 \times 10^{-10} \frac{g_1}{g_2} T^{1/2} \text{ cm}^3 \text{ s}^{-1} \quad (3.243)$$

It is usually weakly dependent on the gas temperature.

3.12 Line absorption and emission

We now describe the absorption of radiation by atoms on the ground state. A bound electron can be promoted to an excited state by absorbing an incoming photon. The corresponding chemical reaction can be written as



The reverse reaction is called spontaneous emission. Because of energy conservation, the absorbed or emitted photons must have $h\nu = E_2 - E_1$. Note that at LTE, because the chemical potential of photons is zero, detailed balance for radiative absorption and emission also gives

$$\mu_\nu + \mu_1 = \mu_2 \text{ or equivalently } \mu_\nu = 0 \text{ and } \mu_1 = \mu_2 \quad (3.245)$$

3.12.1 Emission from a damped harmonic oscillator

The classical treatment of line emission and absorption was due to Lorentz. The idea is to consider the excited state as an harmonic oscillator, where coordinate x represents the orbital radius difference of the electron with respect to the ground state. In this classical view, one writes

$$\ddot{x} + \frac{k}{m_e}x = 0 \text{ where } \omega_{21}^2 = \frac{k}{m_e} \text{ and } \hbar\omega_{21} = E_2 - E_1 \quad (3.246)$$

Without surprise, we obtain the solution of the harmonic oscillator

$$x(t) = x_0 \cos \omega_{21}t \quad (3.247)$$

Using Larmor formula, we compute the power emitted by this accelerated electron as

$$P(t) = \frac{2}{3} \frac{e^2}{c^3} \dot{x}^2 = \frac{2}{3} \frac{e^2}{c^3} \omega_{21}^4 x_0^2 \cos^2(\omega_{21}t) \quad (3.248)$$

As before, we average this power over one period to get the corresponding radiation energy emitted per unit time as

$$P_0 = \frac{1}{3} \frac{e^2 \omega_{21}^4}{c^3} x_0^2 \quad (3.249)$$

We see immediately a problem. Because the initial available energy, namely $E_2 - E_1$ is finite, this radiation cannot last forever. This is in contradiction with the fact that the energy of the harmonic oscillator is conserved. A better model for the excited atom is the *damped harmonic oscillator* whose equation of motion is

$$\ddot{x} + \Gamma \dot{x} + \omega_{21}^2 x = 0 \quad (3.250)$$

We can compute the drag coefficient by imposing that the work done by the friction force per unit time is equal to the radiated power, and this for the original harmonic oscillator. In other words, we neglect the effect of the drag over one oscillator period. This leads to

$$P_{\text{drag}} = -F_{\text{drag}} \dot{x} = m_e \Gamma \dot{x}^2 = m_e \Gamma \omega_{21}^2 x_0^2 \sin^2(\omega_{21}t) \quad (3.251)$$

Here again, we average over one period, and impose that the energy dissipated by friction is equal to the energy radiated away. This gives

$$\frac{1}{2} m_e \Gamma \omega_{21}^2 x_0^2 = \frac{1}{3} \frac{e^2 \omega_{21}^4}{c^3} x_0^2 \quad (3.252)$$

The corresponding drag coefficient writes

$$\boxed{\Gamma = \frac{2}{3} \frac{e^2 \omega_{21}^2}{m_e c^3}} \quad (3.253)$$

This frequency Γ is much smaller than the harmonic oscillator frequency ω_{21} for most photon energies. Indeed, one recognises in the previous equation the electron classical radius r_e

$$\Gamma = \frac{2}{3} \frac{r_e}{c} \omega_{21}^2 \quad (3.254)$$

The corresponding light crossing frequency is very high, with $c/r_e \simeq 10^{21}$ Hz. We see for example for visible light, we have $\omega_{21} \simeq 10^{14}$ Hz, so that $\Gamma \simeq 10^{-7} \omega_{21}$, justifying our previous approach of neglecting the effect of the drag in the harmonic oscillator solution. Using our estimation of the drag coefficient, we now compute the true solution for $x(t)$. We use the Ansatz $x(t) = x_0 \exp(i\omega t)$ with now $\omega \neq \omega_{21}$, and inject in the equation of motion for the damped oscillator. This leads to the dispersion relation

$$-\omega^2 + i\Gamma\omega + \omega_{21}^2 = 0 \quad (3.255)$$

We can solve this second order polynomial equation with

$$\Delta = -\Gamma^2 + 4\omega_{21}^2 \simeq 4\omega_{21}^2 \quad (3.256)$$

because, as we just discussed, the drag frequency is much smaller than the radiation frequency in most cases. The general solution is therefore

$$\omega = \pm\omega_{21} + i\frac{\Gamma}{2} \quad (3.257)$$

Since the orbital coordinate is real and peaks at $t = 0$, we have

$$x(t) = x_0 \exp^{-\Gamma t/2} \cos(\omega_{21} t) \quad (3.258)$$

The corresponding radiated energy, averaged over one period, and using again $\Gamma \ll \omega_{21}$ writes

$$P_0 = \frac{1}{3} \exp^{-\Gamma t} \frac{e^2 \omega_{21}^4}{c^3} x_0^2 \quad (3.259)$$

Integrating in time this power, we get the total radiated energy and we impose it to be equal to the excitation energy

$$E_0 = \frac{1}{3\Gamma} \frac{e^2 \omega_{21}^4}{c^3} x_0^2 = E_2 - E_1 = \hbar\omega_{21} = h\nu_{21} \quad (3.260)$$

This way, we obtain the initial coordinate of the harmonic oscillator x_0 and the entire emission process is described by

$$\boxed{P_0 = h\nu_{21} \Gamma \exp^{-\Gamma t}} \quad (3.261)$$

This model can be used to model the spontaneous emission of light from an excited atom that falls back to the ground state. In order to get the spectrum of this quasi-monochromatic radiation, we need to Fourier transform the orbital motion of the decaying electron.

$$\hat{x}(\omega) = \frac{x_0}{2\pi} \int_0^{+\infty} \frac{1}{2} (\exp^{-i\omega_{21} t} + \exp^{i\omega_{21} t}) \exp^{-\Gamma t/2} \exp^{-i\omega t} dt \quad (3.262)$$

The dominant term is the second one,

$$\hat{x}(\omega) \simeq \frac{x_0}{4\pi} \int_0^{+\infty} \exp[i(\omega_{21}-\omega)-(\Gamma/2)]t \, dt = \frac{x_0}{4\pi} \frac{1}{\Gamma/2 - i(\omega_{21} - \omega)} \quad (3.263)$$

Using Parseval theorem and Larmor formula, we get the radiation spectrum

$$\frac{dW}{d\omega} = \frac{8\pi}{3} \frac{e^2}{c^3} \omega^4 |\hat{x}(\omega)|^2 \quad (3.264)$$

We are mostly interested in the spectral distribution close to the oscillator frequency, so we approximate $\omega \simeq \omega_{21}$. We have

$$\frac{dW}{d\omega} \simeq \frac{1}{2\pi} \frac{1}{3} \frac{e^2 \omega_{21}^4}{c^3} x_0^2 \frac{1}{(\Gamma/2)^2 + (\omega - \omega_{21})^2} \quad (3.265)$$

Injecting the value we determined earlier for x_0 , we find

$$\frac{dW}{d\omega} \simeq \frac{1}{2\pi} h\nu_{21} \frac{\Gamma}{(\Gamma/2)^2 + (\omega - \omega_{21})^2} \quad (3.266)$$

We express the emitted radiation spectrum as a function of ν with $\omega = 2\pi\nu$. We have

$$\frac{dW}{d\omega} = \frac{dW}{2\pi d\nu} \quad (3.267)$$

so that

$$\frac{dW}{d\nu} = h\nu_{21} \frac{\Gamma/4\pi^2}{(\Gamma/4\pi)^2 + (\nu - \nu_{21})^2} \quad (3.268)$$

Finally, since the emitted radiation is isotropic, multiplying by the rate of these deexcitation events, we get

$$j_\nu = \frac{dn_{21}}{dt} \frac{h\nu_{21}}{4\pi} \frac{\Gamma/4\pi^2}{(\Gamma/4\pi)^2 + (\nu - \nu_{21})^2} = \frac{dn_{21}}{dt} \frac{h\nu_{21}}{4\pi} \phi(\nu) \quad (3.269)$$

The spectral energy distribution $\phi(\nu)$ of the emitted radiation is called *the line profile*. In this case, it is the Lorentz profile that describe spontaneous emission due to the radiative decay of the excited level. We still need to estimate the deexcitation rate. It will come later.

3.12.2 Line absorption cross-section

We now move to the description of the interaction of our damped harmonic oscillator with an incoming monochromatic electromagnetic field. The electric field of the planar wave is similar to the Thomson scattering case with

$$E(t) = E_0 \exp^{i\omega t} \quad (3.270)$$

The electron is now subject to the electromagnetic field of the incoming wave, and the equation of motion writes, neglecting the contribution of the magnetic field to the Lorentz force,

$$\ddot{x} + \Gamma\dot{x} + \omega_{21}^2 x = \frac{eE_0}{m_e} \exp^{i\omega t} \quad (3.271)$$

This equation corresponds now a *forced harmonic oscillator*. We are looking for solutions of the form $x(t) = x_0 \exp(i\omega t)$, where ω is the same frequency as the incoming electromagnetic wave. Injecting this into the equation of motion, we obtain

$$(-\omega^2 + i\Gamma\omega + \omega_{21}^2) x_0 = \frac{eE_0}{m_e} \quad (3.272)$$

We find after some manipulations

$$x_0 = \frac{eE_0}{m_e} \frac{\omega_{21}^2 - \omega^2 - i\omega\Gamma}{(\omega_{21}^2 - \omega^2)^2 + \omega^2\Gamma^2} \quad (3.273)$$

The power emitted by the accelerated electron, average over one period, writes

$$P_0 = \frac{1}{3} \frac{e^4 E_0^2}{m_e^2 c^3} \frac{\omega^4}{(\omega_{21}^2 - \omega^2)^2 + \omega^2\Gamma^2} \quad (3.274)$$

We know that the incoming electromagnetic wave has for radiation intensity, using the Poynting vector averaged over one period,

$$S_0 = \frac{c}{8\pi} E_0^2 \quad (3.275)$$

We conclude that the emitted radiation is $P_0 = \sigma(\omega)S_0$ where the absorption cross section is

$$\boxed{\sigma_{21}(\omega) = \sigma_T \frac{\omega^4}{(\omega_{21}^2 - \omega^2)^2 + \omega^2\Gamma^2}} \quad (3.276)$$

We can discuss several regimes for the previous important result.

- For $\omega \rightarrow +\infty$, we see that $\sigma_{21} \rightarrow \sigma_T$. The electromagnetic wave is so energetic that the bound electron behaves like a free electron.
- For $\omega \rightarrow 0$, we have $\sigma_{21} \simeq \sigma_T (\omega^4/\omega_{21}^4)$. This regime is called *Rayleigh scattering*. This is the reason why the sky is blue, as more energetic (blue) photons are absorbed and re-emitted isotropically (in other words, scattered) more than less energetic (red) photons.
- For $\omega \simeq \omega_{21}$, we can Taylor expand the previous equation using $\Delta\omega = \omega - \omega_{21} \ll \omega_{21}$ and obtain the *Lorentz absorption profile*

$$\boxed{\sigma_{21}(\omega) \simeq \sigma_T \frac{\omega_{21}^2}{4(\omega_{21} - \omega)^2 + \Gamma^2}} \quad (3.277)$$

We will use this profile from now on to describe the absorption cross section close to the excitation frequency.

For reasons that will become obvious later, we prefer to use the frequency integrated cross section, called the *equivalent width* of the line

$$\sigma_0 = \int_0^{+\infty} \sigma_{21}(\omega) d\omega \simeq \sigma_T \frac{\omega_{21}^2}{2\Gamma} \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \sigma_T \frac{\pi\omega_{21}^2}{2\Gamma} \quad (3.278)$$

where we used the change of variable

$$x = \frac{2(\omega - \omega_{21})}{\Gamma} \quad (3.279)$$

Injecting the value we found for Γ , we obtain a value independent of the transition frequency

$$\sigma_0 = 2\pi^2 \frac{e^2}{m_e c} \quad (3.280)$$

We can finally write the cross section as $\sigma_{21}(\omega) = \sigma_0\phi(\omega)$, where the line profile is now normalised to 1.

$$\phi(\omega) = \frac{1}{2\pi} \frac{\Gamma}{(\omega_{21} - \omega)^2 + (\frac{\Gamma}{2})^2} \text{ with } \int_0^{+\infty} \phi(\omega)d\omega = 1 \quad (3.281)$$

Note that the absorption cross section is expressed as a function of the wave frequency ω . If we express it as a function of the photon frequency ν , we have instead $\sigma_{21}(\nu) = \sigma_0\phi(\nu)$ with

$$\sigma_0 = \frac{\pi e^2}{m_e c} \text{ and } \phi(\nu) = \frac{\Gamma/4\pi^2}{(\nu_{21} - \nu)^2 + (\Gamma/4\pi)^2} \text{ with } \int_0^{+\infty} \phi(\nu)d\nu = 1 \quad (3.282)$$

The integrated cross section is a constant, with $\sigma_0 \simeq 0.027 \text{ cm}^2 \text{ Hz}$ and independent on the energy level. The shape of the Lorentz line profile has a typical width in photon frequency given by $\Delta\nu = \Gamma/2\pi$. Interestingly, the width of the line profile expressed as a function of the photon wavelength is

$$\Delta\lambda = \frac{\lambda_{21}}{\nu_{21}} \Delta\nu = \frac{c\Gamma}{2\pi\nu_{21}^2} = \frac{4\pi}{3} r_e \quad (3.283)$$

which is the electron classical radius, also independent on the transition frequency. The line can become wider due to the random thermal or turbulent velocities of the emitting atoms. This effect is called Doppler broadening and modifies strongly the line profile but does not change the equivalent width σ_0 . The absorption coefficient can finally be computed as usual by multiplying by the number density of the absorbing atoms, in this case, the population of atoms in the ground state n_1 . Note that this derivation used a semi-classical approach. In order to account for quantum mechanical effect, one introduces a correction factor f_{12} called the *oscillator strength*, which depends on each radiative transition. In conclusion, we have for the absorption coefficient due to radiative excitations from level 1 to level 2

$$\alpha_\nu = n_1 \frac{\pi e^2}{m_e c} f_{12} \phi(\nu) \quad (3.284)$$

Most of the atomic lines are said to be “authorised”, and the oscillator strength is close to 1, with values typically larger than 10^{-2} . A few lines are “forbidden lines”, because of specific quantum mechanical rules. In this case, the oscillator strengths can be very small, with $f_{12} \simeq 10^{-10}$, and the corresponding optical depth also much smaller than that of a normal line. Molecular lines have typical oscillator strength $f_{12} \simeq 10^{-5}$. This is because the effective charge that enters Larmor formula is much smaller for most molecules. For H_2 , for example, the dipole moment that measures the strength of the Larmor emission is zero. For CO , the asymmetry of the charge distribution leads to a larger dipole moment, but still two to three orders of magnitude smaller than the one of the first excited level of the Hydrogen atom.

3.13 Einstein relations

We have seen in the previous sections that the emissivity of the spontaneous radiative decay of the atomic excited states depends on the unknown rate of deexcitation. This rate is encoded by the *first Einstein coefficient* noted A_{21} so that

$$\frac{dn_{21}}{dt} = n_2 A_{21} \quad (3.285)$$

The emissivity coefficient then writes

$$j_\nu = n_2 \frac{h\nu_{21}}{4\pi} A_{21} \phi(\nu) \quad (3.286)$$

Following the same convention, Einstein introduced the *second Einstein coefficient* in the absorption coefficient as

$$\alpha_\nu = n_1 \frac{h\nu_{21}}{4\pi} B_{12} \phi(\nu) \quad (3.287)$$

He also introduced the *third Einstein coefficient* to describe induced (or stimulated) emission, a process we have already discussed and associated to the Bose enhancement factor.

$$j_\nu^{\text{induced}} = n_2 \frac{h\nu_{21}}{4\pi} B_{21} I_\nu \phi(\nu) \quad (3.288)$$

We already know that this term is associated to the quantum correction of the spontaneous emission

$$j_\nu = n_2 \frac{h\nu_{21}}{4\pi} A_{21} \phi(\nu) (1 + \mathcal{N}_\nu) \quad (3.289)$$

We already know from our previous semi-classical derivation that the different line profiles are all given by the same Lorentz profile. We also know that the second Einstein coefficient is given by

$$\frac{h\nu_{21}}{4\pi} B_{12} = \frac{\pi e^2}{m_e c} f_{12} \quad (3.290)$$

We now derive the famous Einstein relations. If the radiation and the fluid are both at LTE, then we have to fulfill 3 conditions. First, the level population are given by the Boltzmann relation we have derived earlier

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp^{-(E_2-E_1)/k_B T} = \frac{g_2}{g_1} \exp^{-h\nu_{21}/k_B T} \quad (3.291)$$

Second, emission and absorption must balance each other perfectly.

$$j_\nu + j_\nu^{\text{induced}} = \alpha_\nu I_\nu \quad (3.292)$$

or equivalently

$$n_2 \frac{h\nu_{21}}{4\pi} A_{21} \phi(\nu) + n_2 \frac{h\nu_{21}}{4\pi} B_{21} I_\nu \phi(\nu) = n_1 \frac{h\nu_{21}}{4\pi} B_{12} I_\nu \phi(\nu) \quad (3.293)$$

Third, the radiation specific intensity must be equal to the black body spectrum

$$I_\nu = B_\nu(T) \quad (3.294)$$

The second equation simplifies into

$$n_2 (A_{21} + B_{21} B_\nu(T)) = n_1 B_{12} B_\nu(T) \quad (3.295)$$

and finally we get

$$n_2 A_{21} = (n_1 B_{12} - n_2 B_{21}) B_\nu(T) \quad (3.296)$$

which leads to

$$\frac{A_{21}/B_{21}}{(n_1 B_{12}/n_2 B_{21} - 1)} = B_\nu(T) = \frac{2h\nu_{21}^3/c^2}{\exp(h\nu_{21}/k_B T) - 1} \quad (3.297)$$

Note that we have evaluated the black body spectrum at the exact line position. Since the left equation must match the right one, we deduce the following two relations between the Einstein coefficients.

$$\boxed{A_{21} = \frac{2h\nu_{21}^3}{c^2} B_{21}} \quad (3.298)$$

and

$$\boxed{g_1 B_{12} = g_2 B_{21}} \quad (3.299)$$

These are known as *the Einstein relations*. The first relation is indeed nothing else than the Bose enhancement factor multiplied to the spontaneous emission. As discussed before, induced emission can be considered as negative absorption. Let's assume we know the second Einstein coefficient using the semi-classical theory.

$$B_{12} = \frac{4\pi^2 e^2}{m_e c h \nu_{21}} f_{12} \quad (3.300)$$

We deduce the third coefficient

$$B_{21} = \frac{g_1}{g_2} B_{12} = \frac{4\pi^2 e^2}{m_e c h \nu_{21}} \frac{g_1}{g_2} f_{12} \quad (3.301)$$

and the first coefficient

$$A_{21} = \frac{2h\nu_{21}^3}{c^2} B_{21} = \frac{8\pi^2 e^2 \nu_{21}^2}{m_e c^3} \frac{g_1}{g_2} f_{12} = 3\Gamma \frac{g_1}{g_2} f_{12} \quad (3.302)$$

We can find in astrophysical database the first Einstein coefficient, in units of $[s^{-1}]$. We then have to deduce the second and third using the previous relations. We can also find the oscillator strength f_{21} which defines the second Einstein coefficient. We then deduce the first and third using the previous relations. Using the semi-classical theory, we find for visible light with $\nu_{21} \simeq 10^{14}$ Hz that $\Gamma \simeq 2.4 \times 10^6 \text{ sec}^{-1}$. This frequency is associated to the inverse of the life time of the excited state, which we find to be for visible light very short, namely around 4×10^{-7} s. For sub-mm wavelengths associated to molecular lines, the lifetime is much longer, of the order of several years, because of the much smaller oscillator strength. Although we derived the Einstein relations under LTE conditions, since the various coefficients are microscopic properties of the atoms, we conclude that these relations are true even in the non-LTE case. In the general case, the intensity is not necessarily a black body, the level populations do not follow the Boltzmann relations, and emission does not strictly balance absorption, but the Einstein relations still holds. We can summarise our results by writing the emissivity coefficient as

$$j_\nu = \frac{\pi e^2}{m_e c} f_{12} \frac{2h\nu_{21}^3}{c^2} \frac{g_2}{g_1} n_2 \phi(\nu) \quad (3.303)$$

and the absorption coefficient as

$$\alpha_\nu = \frac{\pi e^2}{m_e c} f_{12} \left(n_1 - \frac{g_2}{g_1} n_2 \right) \phi(\nu) \quad (3.304)$$

It is left as an exercise to the reader to show that under LTE conditions, the two previous quantities satisfy Kirchhoff's theorem. Under most conditions, the second level is much less populated than the ground state. The absorption coefficient remains therefore positive. But in some rare cases, we can have an inversion of the level population. The resulting negative absorption coefficient leads to the laser or maser effect.

3.14 Non-LTE level population

We have determined the absorption rates of the radiative reactions leading to the excitation of an atom from level 1 to level 2, and the emission rate of the radiative reactions leading to the deexcitation of an atom from level 2 to level 1. These radiative reaction rates can now be

added to the collision reactions rates we have already presented. We have already dealt with spontaneous emission, for which

$$\frac{dn_{21}}{dt} = n_2 A_{21} \quad (3.305)$$

owing to the definition of the first Einstein coefficient. We then want to compute the number of transitions from 1 to 2 due to absorption of the incoming radiation. We just need to divide the absorbed energy by the energy of one photon, and then integrate over all solid angles and all frequencies. This gives

$$\frac{dn_{12}}{dt} = \int_0^{+\infty} \int_{4\pi} \alpha_\nu \frac{I_\nu}{h\nu} d\Omega d\nu \quad (3.306)$$

where

$$\alpha_\nu = (n_1 B_{12} - n_2 B_{21}) \frac{h\nu_{21}}{4\pi} \phi(\nu) \quad (3.307)$$

Since the line profile is very narrow at $\nu \simeq \nu_{21}$, we have

$$\frac{dn_{12}}{dt} \simeq (n_1 B_{12} - n_2 B_{21}) \int_0^{+\infty} \phi(\nu) d\nu \frac{1}{4\pi} \int_{4\pi} I_\nu d\Omega \quad (3.308)$$

We recognise the mean radiation intensity J_ν , which is then weighted by the line profile and integrated over frequency. We define

$$\bar{J}_{21} = \int_0^{+\infty} J_\nu \phi(\nu) d\nu \quad (3.309)$$

We can finally add this up in the rate equation for level 1 as

$$\boxed{\frac{dn_1}{dt} = n_2 C_{21} - n_1 C_{12} + n_2 A_{21} - (n_1 B_{12} - n_2 B_{21}) \bar{J}_{21} = -\frac{dn_2}{dt}} \quad (3.310)$$

We have seen that the first Einstein coefficient is usually quite large, with corresponding life times much smaller than the typical astrophysical time scales. In this case, we can safely assume that the system will reach chemical equilibrium, which can be written as

$$n_2 (C_{21} + A_{21} + B_{21} \bar{J}_{21}) = n_1 (C_{12} + B_{12} \bar{J}_{21}) \quad (3.311)$$

We conclude that in the general case $J_\nu \neq B_\nu(T)$, we don't get the LTE level population, which can be defined by $n_2 C_{21} = n_1 C_{12}$. Using the definitions $C_{21} = n_e \langle \sigma_{21} v \rangle$ and $C_{12} = n_e \langle \sigma_{12} v \rangle$, we see that we converge towards the LTE limit only if the electron density is high enough, namely

$$n_e \gg n_{\text{crit}} = \frac{A_{21} + B_{21} \bar{J}_{21}}{\langle \sigma_{21} v \rangle} \quad (3.312)$$

This critical density depends on the radiation field, which makes the overall coupled problem of finding the level population and the corresponding radiation field a daunting task in general. This requires complex iterative numerical methods and is still a field on intense research. Note that when the level population is not at LTE, it is customary to define the *excitation temperature* as

$$k_B T_{\text{exc}} = h\nu_{12} / \ln \frac{n_1 g_2}{n_2 g_1} \quad (3.313)$$

Only at LTE do we recover $T_{\text{exc}} = T$. The stronger the radiation field, the larger the deviation from LTE. The most optimistic case is to consider $\bar{J}_{21} = 0$. This gives

$$n_{\text{crit}} = \frac{A_{21}}{\langle \sigma_{21} v \rangle} \quad (3.314)$$

For atomic lines, we have $A_{21} \simeq 10^7$ Hz. We have also computed previously the collisional deexcitation cross section as

$$\langle \sigma_{21v} \rangle = 3.9 \times 10^{-10} \frac{g_1}{g_2} T^{1/2} \text{ cm}^3 \text{ s}^{-1} \quad (3.315)$$

We therefore find for the critical density

$$n_{\text{crit}} \simeq 2.5 \times 10^{16} T^{-1/2} \text{ cm}^{-3} \quad (3.316)$$

This is the typical density of the Sun photosphere. LTE conditions can thus be found mostly in stellar interiors. In all the other environments, we are in what is called *coronal equilibrium*. In this case, we see that chemical equilibrium writes as

$$n_2 = n_1 n_e \frac{\langle \sigma_{12v} \rangle}{A_{21}} \quad (3.317)$$

In this regime, line emission is therefore proportional to n^2 . At LTE, on the other hand, line emission is proportional to n . For molecular lines, we have $A_{21} \simeq 10^{-7}$ Hz. Using the same cross section, we find $n_{\text{crit}} \simeq 10^3 \text{ cm}^{-3}$. This corresponds to typical conditions inside dense molecular clouds, in which modelling properly the transition from non-LTE to LTE is crucial.

3.15 Ionisation and recombinaison

This last section will be dedicated to bound-free radiation. This corresponds to inelastic collisions leading to the absorption of a energetic enough photon and the ejection of a bound electron into the continuum, or to the capture of a free electron from the continuum and the emission of a photon. We can write these chemical reactions as



Note that the bound electron can be in any of the previously discussed bound states. For simplicity, however, we assume that the level population of the neutral Hydrogen atom is entirely in the ground state.

3.15.1 Saha relation

In LTE, we can write the relation between the chemical potential. Using the fact that $\mu_\nu = 0$, we have

$$\mu_0 = \mu_e + \mu_+ \quad (3.319)$$

For each of the three species, at LTE, the distribution function follows the Maxwell-Boltzmann with

$$f_0 = \frac{g_0}{h^3} \exp^{\mu_0/k_B T} \exp^{-E_1/k_B T} \exp^{-\frac{p_0^2}{2m_p}/k_B T} \quad (3.320)$$

$$f_+ = \frac{g_+}{h^3} \exp^{\mu_+/k_B T} \exp^{-\frac{p_+^2}{2m_p}/k_B T} \quad (3.321)$$

$$f_e = \frac{2}{h^3} \exp^{\mu_e/k_B T} \exp^{-\frac{p_e^2}{2m_e}/k_B T} \quad (3.322)$$

Integrating over momentum space leads to

$$n_{H^0}(\mathbf{x}, t) = \int_{\mathbb{R}^3} f_0(\mathbf{x}, \mathbf{p}_0, t) d^3 p_0 = \frac{g_0}{h^3} \exp^{-E_1/k_B T} \exp^{\mu_0/k_B T} \int_{\mathbb{R}^3} \exp^{-\frac{p_0^2}{2m_p}/k_B T} d^3 p_0 \quad (3.323)$$

We can integrate the Gaussian, leading to

$$n_{\text{H}^0}(\mathbf{x}, t) = \frac{g_0}{h^3} \exp^{-E_1/k_B T} \exp^{\mu_0/k_B T} (2\pi m_p k_B T)^{3/2} \quad (3.324)$$

We can write a similar equation for the number densities of the free electrons and of the naked protons with

$$n_{\text{H}^+}(\mathbf{x}, t) = \frac{g_+}{h^3} \exp^{\mu_+/k_B T} (2\pi m_p k_B T)^{3/2} \quad (3.325)$$

and

$$n_{e^-}(\mathbf{x}, t) = \frac{2}{h^3} \exp^{\mu_e/k_B T} (2\pi m_e k_B T)^{3/2} \quad (3.326)$$

The additive relation between the chemical potential forces us to combine the three number densities into

$$\boxed{\frac{n_{\text{H}^+} n_{e^-}}{n_{\text{H}^0}} = \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} \frac{2g_+}{g_0} \exp^{-\chi_I/k_B T}} \quad (3.327)$$

where we introduced the ionisation potential for the ground state $\chi_I = -E_1 = 13.6$ eV. Note that $E_1 < 0$ and $\chi_I > 0$. This relation, known as Saha relation, together with charge neutrality and proton conservation allows us to determine the ionisation state of the gas at LTE.

$$n_{\text{H}^+} = n_{e^-} \quad \text{and} \quad n_{\text{H}} = \frac{\rho}{m_p} = n_{\text{H}^+} + n_{\text{H}^0} \quad (3.328)$$

3.15.2 Einstein-Milne relation

The previous chemical reactions lead to the absorption or the emission of a photon. Energy conservation during the inelastic collision writes

$$\frac{1}{2} m_e v_e^2 + \frac{1}{2} m_p v_+^2 = h\nu + \frac{1}{2} (m_p + m_e) v_0^2 + E_1 \quad (3.329)$$

Because the proton is so much more massive than the electron, the centre of mass velocity can be considered equal to the proton and neutral Hydrogen velocity, while the relative velocity is equal to the electron velocity.

$$\mathbf{v}_+ = \mathbf{v}_0 = \mathbf{V} \quad \text{and} \quad \mathbf{v}_e = \mathbf{v} \quad (3.330)$$

We then obtain the collision invariant, valid both for the emission or the absorption processes,

$$\frac{1}{2} m_e v^2 = h\nu - \chi_I \quad (3.331)$$

We define the absorption cross-section, called the *photo-ionisation cross-section* $\sigma_{\text{ion}}(\nu)$, so that the number of collisions per unit time and per unit volume between neutral Hydrogens and incoming photons leading to by a photo-ionisation event is

$$dN_{\text{ion}} = \sigma_{\text{ion}}(\nu) c f_\nu f_0 d^3 p_\nu d^3 p_0 dt d^3 x \quad (3.332)$$

We define the emission cross-section, called the *recombination cross-section* $\sigma_{\text{rec}}(v)$, so that the number of collisions per unit time and per unit volume between protons and free electrons leading to a recombination event is

$$dN_{\text{rec}} = \sigma_{\text{rec}}(v) v f_e f_+ (1 + \mathcal{N}_\nu) d^3 p_e d^3 p_+ dt d^3 x \quad (3.333)$$

Note that in the last equation, we have properly multiply the cross-section by the Bose enhancement factor for photons, which leads to the induced emission process for recombination. We

now write the detailed balance condition for LTE, with $dN_{\text{ion}} = dN_{\text{rec}}$. We already know that $d^3p_+ = d^3p_0$, From energy conservation, we also get

$$m_e v dv = h d\nu \quad (3.334)$$

The photon and electron momentum are defined as

$$p_\nu = \frac{h\nu}{c} \text{ and } p_e = m_e v \quad (3.335)$$

At LTE, we also know that the photon distribution function follows the black-body spectrum, with

$$f_\nu = \frac{2}{h^3} \mathcal{N}_\nu \text{ and } \frac{\mathcal{N}_\nu}{1 + \mathcal{N}_\nu} = \exp^{-h\nu/k_B T} \quad (3.336)$$

Injecting the various LTE distribution functions in the detailed balance equation leads to

$$\sigma_{\text{ion}}(\nu) c g_0 d^3 p_\nu = \sigma_{\text{rec}}(v) v g_+ d^3 p_e \quad (3.337)$$

Using $m_e v dv = h d\nu$ together with

$$d^3 p_\nu = \frac{h^3}{c^3} \nu^2 d\nu d\Omega \text{ and } d^3 p_e = m_e^3 v^2 dv d\Omega \quad (3.338)$$

we find the Einstein-Milne relation

$$\boxed{\sigma_{\text{ion}}(\nu) \frac{h^2 \nu^2}{c^2} g_0 = \sigma_{\text{rec}}(v) m_e^2 v^2 g_+} \quad (3.339)$$

We have derived this relation at LTE, but since both cross section are microscopic properties of the particles, they are valid all the time, even far from equilibrium. Interestingly, if we derive the recombination cross section $\sigma_{\text{rec}}(v)$, we can deduce immediately the photo-ionisation cross section $\sigma_{\text{ion}}(\nu)$.

3.16 Bound-free radiation

We now write the absorbed energy due to photo-ionisation events by multiplying the number of events by the absorbed photon energy $h\nu$ and integrate over Hydrogen atoms momentum space

$$dE_{\text{ion}} = n_{\text{H}^0} h\nu \sigma_{\text{ion}}(\nu) c f_\nu d^3 p_\nu dt d^3 x = n_{\text{H}^0} \sigma_{\text{ion}}(\nu) \frac{h^4 \nu^3}{c^2} f_\nu d\nu d\Omega dt d^3 x \quad (3.340)$$

We recognise the specific intensity

$$I_\nu = \frac{h^4 \nu^3}{c^2} f_\nu \quad (3.341)$$

We deduce that the absorption coefficient for photoionisation is

$$\alpha_\nu = n_{\text{H}^0} \sigma_{\text{ion}}(\nu) \quad (3.342)$$

As before, we consider the induced emission as a negative absorption. We can do this thanks to Einstein-Milne relation. Indeed, the energy emitted by induced emission during recombination events can be written, after multiplying by the photon energy $h\nu$ and integrating over protons momentum space

$$dE_{\text{rec}}^{\text{induced}} = n_{\text{H}^+} h\nu \sigma_{\text{rec}}(v) v f_e \mathcal{N}_\nu d^3 p_e dt d^3 x = n_{\text{H}^+} h\nu \sigma_{\text{rec}}(v) v f_e \mathcal{N}_\nu m_e^3 v^2 dv d\Omega dt d^3 x \quad (3.343)$$

We use the connection between the free electron and the emitted photon with

$$\frac{1}{2}m_e v^2 = h\nu - \chi_I \text{ and } m_e v dv = h d\nu \quad (3.344)$$

and assume that electrons are in LTE, but not photons, so that

$$f_e = \frac{n_e}{(2\pi m_e k_B T)^{3/2}} \exp^{-\frac{1}{2}m_e v^2/k_B T} \quad (3.345)$$

Injecting this into the induced emission energy and using the Einstein-Milne relation, we get

$$dE_{\text{rec}}^{\text{induced}} = \frac{g_0}{g_+} \frac{n_{\text{H}^+} n_e \exp^{\chi_I/k_B T}}{(2\pi m_e k_B T)^{3/2}} \sigma_{\text{ion}}(\nu) \exp^{-h\nu/k_B T} \frac{h^4 \nu^3}{c^2} \mathcal{N}_\nu d\nu d\Omega dt d^3x \quad (3.346)$$

Assuming that the ionisation state is at LTE, we can use Saha relation and we get

$$dE_{\text{rec}}^{\text{induced}} = n_{\text{H}^0} \frac{2}{h^3} \sigma_{\text{ion}}(\nu) \exp^{-h\nu/k_B T} \frac{h^4 \nu^3}{c^2} \mathcal{N}_\nu d\nu d\Omega dt d^3x \quad (3.347)$$

$$dE_{\text{rec}}^{\text{induced}} = n_{\text{H}^0} \sigma_{\text{ion}}(\nu) \exp^{-h\nu/k_B T} \frac{h^4 \nu^3}{c^2} f_\nu d\nu d\Omega dt d^3x \quad (3.348)$$

We recognise again I_ν . Adding this, with a negative sign, to the absorption coefficient, we finally get

$$\alpha_\nu = n_{\text{H}^0} \sigma_{\text{ion}}(\nu) \left(1 - \exp^{-h\nu/k_B T}\right) \quad (3.349)$$

Note that if electrons are at LTE, but the ionisation state is not at LTE, then we cannot perform the last simplification. In the general case, the absorption coefficient writes

$$\boxed{\alpha_\nu = n_{\text{H}^0} \sigma_{\text{ion}}(\nu) \left(1 - \frac{n_{\text{H}^0}^* \exp^{-h\nu/k_B T}}{n_{\text{H}^0}}\right)} \quad (3.350)$$

where the * value refer to the H^0 density computed assuming LTE. It can be obtained by inverting Saha relation as

$$n_{\text{H}^0}^* = n_{\text{H}^+} n_{e^-} \frac{g_0}{2g_+} \left(\exp^{\chi_I/k_B T} \left/ \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} \right. \right) \quad (3.351)$$

Because this inverse Saha relation is quite complicated, it is customary to write it as

$$\boxed{n_{\text{H}^0}^* = n_{\text{H}^+} n_{e^-} f(T)} \quad (3.352)$$

In order to compute the emissivity coefficient of recombination radiation, we will use the trick of invoking Kirchoff's relation, which is valid if the gas is at LTE, namely

$$j_\nu = \alpha_\nu B_\nu(T) \quad (3.353)$$

Since again the microscopic properties of the gas are valid also outside of LTE, we get for the emissivity coefficient

$$\boxed{j_\nu = n_{\text{H}^0}^* \sigma_{\text{ion}}(\nu) \frac{2h\nu^3}{c^2} \exp^{-h\nu/k_B T}} \quad (3.354)$$

where again the * symbol refers to the value of the neutral Hydrogen density calculated assuming that Saha relation holds, which is different than the true n_{H^0} .

3.17 Recombination and photo-ionisation cross sections

We follow here the semi-classical derivation proposed by Kramers in 1923. The scenario we present now is based on a free electron approaching a naked proton at rest on a Coulomb orbit with impact parameter b and velocity v . We have already computed the emitted power to derive the emissivity of Bremsstrahlung. We found the spectrum due to one single electron to be being decelerated by the proton

$$\frac{dW}{d\omega} = \frac{8}{3\pi} \frac{e^6}{c^3 m_e^2 b^2 v^2} \text{ for } \omega \leq \frac{v}{b} \quad (3.355)$$

and zero otherwise. Most of the electrons will just walk by without being captured by the proton, and contribute to the free-free, Bremsstrahlung radiation. We are interested in those electrons that will land in one of the bound states of the Hydrogen atom. Since the classical spectrum is flat, we can identify each photon emitted by an electron captured by level n with the band of energy in between two consecutive bound states. The highest energy photon will correspond to a recombination to the ground state, while lower and lower energy photons will correspond to a recombination to less and less bound states. Following Kramers' original idea, we write each energy level as

$$E_n = -\frac{\chi_I}{n^2} \quad (3.356)$$

The frequency band between two consecutive states is

$$\Delta\omega = \frac{\chi_I}{\hbar} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \simeq \frac{\chi_I}{\hbar} \frac{2}{n^3} \quad (3.357)$$

The cross section for the free electrons captured to a given bound state can be obtained by requiring

$$\frac{dW}{d\omega} \Delta\omega = h\nu \quad (3.358)$$

We use for the ionising potential

$$\chi_I = \frac{1}{2} \frac{e^2}{r_B} = \frac{m_e e^4}{2\hbar^2} \text{ with } r_B = \frac{\hbar^2}{m_e e^2} \quad (3.359)$$

We finally obtain using $\sigma_{\text{rec}}(v) = \pi b^2$

$$\sigma_{\text{rec}}(v) = \frac{64\pi^3}{3} \frac{e^{10}}{c^3 h^4 m_e} \frac{1}{n^3 \nu v^2} \quad (3.360)$$

Using the Einstein-Milne relation with $g_0/g_+ = n^2$, we immediately obtain the photoionisation cross section as

$$\sigma_{\text{ion}}(\nu) = \frac{64\pi^3}{3} \frac{e^{10} m_e}{c h^6} \frac{1}{n^5 \nu^3} \quad (3.361)$$

Here again, like for Bremsstrahlung, we can introduce a Gaunt factor g_{bf} to account for possible quantum correction. Like for Bremsstrahlung, a factor $\pi/\sqrt{3}$ can then be introduced to account for the exact shape of the acceleration profile. We have now computed the photo-ionisation cross section and can therefore deduce the corresponding values for α_ν and j_ν , as explained in the previous section. We leave to the reader as an exercise the direct computation of the emissivity coefficient using the recombination cross section, without invoking Kirchhoff's law. The result should be identical.

We now use the photo-ionisation cross section to compute recombination cooling. We restrict ourselves to recombinations to the ground state with $n = 1$.

$$\mathcal{C}_{\text{rec}} = \int_{\chi_I}^{+\infty} 4\pi j_\nu d\nu = 4\pi n_{\text{H}^+} n_{\text{e}^-} f(T) \int_{\chi_I}^{+\infty} \sigma_{\text{ion}}(\nu) \frac{2h\nu^3}{c^2} \exp^{-h\nu/k_B T} \quad (3.362)$$

We see that the $1/\nu^3$ dependence of the cross-section cancels out with the ν^3 dependence of the black body spectrum. We are left only with a simple integral over the exponential. Injecting the function $f(T)$ from the Saha relation, we obtain

$$\mathcal{C}_{\text{rec}} = n_{\text{H}^+} n_{\text{e}^-} \frac{128\pi^3}{3} \frac{e^{10}}{c^3 h^3} (2\pi m_e k_B T)^{-1/2} \quad (3.363)$$

If we multiply by $\pi/\sqrt{3}$ to account for the Gaunt factor, we obtain finally

$$\boxed{\mathcal{C}_{\text{rec}} = 2.24 \times 10^{-22} g_{\text{bf}} n_{\text{H}^+} n_{\text{e}^-} T^{-0.5}} \quad (3.364)$$

If we compare this value to Bremsstrahlung cooling, we find that Hydrogen recombination cooling dominates only for $T < 10^5$ K.

3.18 Non-LTE recombination and ionisation

In order to compute the ionisation state of the gas when it is not in LTE, we need to derive rate equations for the ionisation/recombination balance. A very important chemical reaction needs to be introduced first, namely *collisional ionisation*. It features a free electron that collides with a neutral Hydrogen atom and manages to eject the bound electron into the continuum. It can be written as follows



Note that it does not involve any photon. It is a pure collisional process. The reverse reaction is called *three-body recombination*. In this case, the energy released by the capture of a free electron is not provided to the system as radiation, but as an increase in the kinetic energy of the second collision partner. This process is quite rare at low density, because it requires to bring together 3 particles instead of 2 for binary collisions. At high density, however, it is of primary importance, and in the absence of radiation, this is the process that will drive the system towards LTE. We can now enumerate the 5 processes involved in the detailed ionisation balance.

- photo-ionisation
- radiative recombination
- stimulated radiative recombination
- collisional ionisation
- three-body collisional recombination

We now compute each reaction rate one by one.

3.18.1 Photo-ionisation rate

The total number of photo-ionisation is obtained by integrating the absorbed radiation over angle and frequency. Only photons with $h\nu > \chi_I$ will contribute to photo-ionisations.

$$\frac{dn_{\text{H}^0}}{dt} = - \int_{\chi_I}^{+\infty} \int_{4\pi} \alpha_\nu \frac{I_\nu}{h\nu} d\nu d\Omega \quad (3.366)$$

where we divided by $h\nu$ to get the number of photon absorbed per unit time. We use for α_ν our previous result, which leads to

$$\frac{dn_{\text{H}^0}}{dt} = -n_{\text{H}^0} \int_{\chi_I}^{+\infty} \int_{4\pi} \sigma_{\text{ion}}(\nu) \frac{I_\nu}{h\nu} d\nu d\Omega = -n_{\text{H}^0} \Gamma_{\text{ion}} \quad (3.367)$$

where the photo-ionisation rate is traditionally used and defined as

$$\Gamma_{\text{ion}} = \int_{\chi_I}^{+\infty} 4\pi \sigma_{\text{ion}}(\nu) \frac{J_\nu}{h\nu} d\nu \quad (3.368)$$

where we replaced the specific intensity by the angle average intensity J_ν .

3.18.2 Radiative and stimulated recombination rates

We now compute the radiative recombination rate. We have seen earlier that it was described by the emissivity coefficient

$$j_\nu = n_{\text{H}^0}^* \sigma_{\text{ion}}(\nu) \frac{2h\nu^3}{c^2} \exp^{-h\nu/k_{\text{B}}T} \quad (3.369)$$

where $n_{\text{H}^0}^* = n_{\text{H}} + n_{\text{e}} - f(T)$. The recombination rate is just the total number of radiative recombinations, leading to

$$\frac{dn_{\text{H}^0}}{dt} = \int_{\chi_I}^{+\infty} 4\pi \frac{j_\nu}{h\nu} d\nu = n_{\text{H}} + n_{\text{e}} - f(T) \int_{\chi_I}^{+\infty} 4\pi \sigma_{\text{ion}}(\nu) \frac{2\nu^2}{c^2} \exp^{-h\nu/k_{\text{B}}T} d\nu \quad (3.370)$$

It is customary to write this as

$$\frac{dn_{\text{H}^0}}{dt} = n_{\text{H}} + n_{\text{e}} - \alpha_{\text{rec}}(T) \quad (3.371)$$

with

$$\alpha_{\text{rec}}(T) = f(T) \int_{\chi_I}^{+\infty} 4\pi \sigma_{\text{ion}}(\nu) \frac{2\nu^2}{c^2} \exp^{-h\nu/k_{\text{B}}T} d\nu \quad (3.372)$$

A good approximation for the recombination rate is to divide the recombination cooling rate by the ionisation potential $\chi_I = 13.6$ eV. This leads to

$$\alpha_{\text{rec}}(T) \simeq 1.02 \times 10^{-11} T^{-0.5} \text{cm}^3 \text{s}^{-1} \quad (3.373)$$

The stimulated recombination coefficient can be computed following a similar procedure. Using the corresponding cross section derived before, we have

$$\frac{dn_{\text{H}^0}}{dt} = \int_{\chi_I}^{+\infty} \int_{4\pi} \alpha_\nu \frac{I_\nu}{h\nu} d\nu d\Omega \quad (3.374)$$

with

$$\alpha_\nu = n_{\text{H}^0}^* \sigma_{\text{ion}}(\nu) \exp^{-h\nu/k_{\text{B}}T} \quad (3.375)$$

This leads to

$$\frac{dn_{\text{H}^0}}{dt} = n_{\text{H}^+} n_{e^-} f(T) \int_{\chi_I}^{+\infty} \int_{4\pi} \sigma_{\text{ion}}(\nu) \exp^{-h\nu/k_{\text{B}}T} \frac{I_\nu}{h\nu} d\nu d\Omega \quad (3.376)$$

We write this as

$$\frac{dn_{\text{H}^0}}{dt} = n_{\text{H}^+} n_{e^-} \Gamma_{\text{rec}}(T) \quad (3.377)$$

where

$$\Gamma_{\text{rec}}(T) = f(T) \int_{\chi_I}^{+\infty} 4\pi \sigma_{\text{ion}}(\nu) \exp^{-h\nu/k_{\text{B}}T} \frac{J_\nu}{h\nu} d\nu \quad (3.378)$$

3.18.3 Collisional ionisation and three-body recombination rates

The last two rates are computed using the newly presented chemical reactions, namely ionisation and recombination featuring no photon but only free electron. These are the equivalent of the collisional excitations and deexcitations for line transitions. We write the collisional ionisation rate between a free electron and an Hydrogen atom as

$$\frac{dn_{\text{H}^0}}{dt} = -n_{\text{H}^0} n_{e^-} \langle \sigma_{\text{ion}} v \rangle = -n_{\text{H}^0} n_{e^-} \alpha_{\text{ion}}(T) \quad (3.379)$$

Note that this is not the photo-ionisation cross section $\sigma_{\text{ion}}(\nu)$ that depends on photon frequency. This one depends on the electron velocity $\sigma_{\text{ion}}(v)$. Usually, this collision cross section is similar to the elastic cross section we have computed for a Coulomb gas, times a small probability computed in a quantum mechanical framework. Many astrophysical databases provide tabulated values for this $\alpha_{\text{ion}}(T)$. We can estimate it using a simple hard sphere model with constant cross section $\sigma_0 \simeq 10^{-15} \text{ cm}^2$, together with the requirement that the electron energy has to be larger than the ionisation potential to be able to ionise the atom. This leads to

$$n_e \langle \sigma_{\text{ion}} v \rangle \simeq \sigma_0 \int_{\chi_I}^{+\infty} v f_e(v) dv \simeq n_e \sigma_0 \left(\frac{k_{\text{B}}T}{m_e} \right)^{1/2} \exp^{-\chi_I/k_{\text{B}}T} \quad (3.380)$$

We finally obtain the collisional ionisation cross section

$$\alpha_{\text{ion}}(T) = \sigma_0 \left(\frac{k_{\text{B}}T}{m_e} \right)^{1/2} \exp^{-\chi_I/k_{\text{B}}T} \quad (3.381)$$

The numerical value can be estimated as

$$\alpha_{\text{ion}} = 3.9 \times 10^{-10} T^{1/2} \exp^{-\chi_I/k_{\text{B}}T} \quad (3.382)$$

The reverse reaction, three-body recombination, is more tricky to compute. For the time being, we write it formally as a three body collision, using

$$\frac{dn_{\text{H}^0}}{dt} = n_{\text{H}^+} n_{e^-}^2 \delta_{\text{rec}}(T) \quad (3.383)$$

Note that we don't need to compute this rate from first principle. We will use the same argument as before, requiring that at LTE, if the electron density is high enough, we can neglect all the other processes and write at chemical equilibrium

$$\frac{dn_{\text{H}^0}}{dt} = 0 \text{ so that } n_{\text{H}^+} n_{e^-}^2 \delta_{\text{rec}}(T) = n_{\text{H}^0} n_{e^-} \alpha_{\text{ion}}(T) \quad (3.384)$$

We see that we have the relation

$$\frac{n_{\text{H}^+}n_{\text{e}^-}}{n_{\text{H}^0}} = \frac{\alpha_{\text{ion}}(T)}{\delta_{\text{rec}}(T)} = \left(\frac{2\pi m_e k_{\text{B}} T}{h^2} \right)^{3/2} \frac{2g_+}{g_0} \exp^{-\chi_1/k_{\text{B}}T} \quad (3.385)$$

where the right-hand side comes from Saha relation. Since the two coefficients are microscopic properties of the plasma, this relation is valid at all time, not only at LTE. If we know α_{ion} , we can therefore deduce δ_{rec} . Using the just derived form for the collisional ionisation coefficient, we find for the ground state $n = 1$

$$\delta_{\text{rec}}(T) = \frac{\sigma_0 h^3}{\sqrt{8\pi} m_e} (2\pi m_e k_{\text{B}} T)^{-1} \quad (3.386)$$

Using our simple hard sphere model, we found the following numerical form

$$\delta_{\text{rec}} = 2.02 \times 10^{-25} T^{-1} \text{ cm}^6 \text{ s}^{-1} \quad (3.387)$$

3.18.4 Ionisation rate equation

We can now combine these 5 processes into a rate equation for the formation or destruction of H^0 , with

$$\frac{dn_{\text{H}^0}}{dt} = \delta_{\text{rec}}(T)n_{\text{H}^+}n_{\text{e}^-}^2 + n_{\text{H}^+}n_{\text{e}^-}\alpha_{\text{rec}}(T) + n_{\text{H}^+}n_{\text{e}^-}\Gamma_{\text{rec}}(T) - \alpha_{\text{ion}}(T)n_{\text{H}^0}n_{\text{e}^-} - n_{\text{H}^0}\Gamma_{\text{ion}} \quad (3.388)$$

We see that the first 3 terms are all recombination processes. At high density, they will be dominated by the three-body term because it scales as n^3 , while the others scale as n^2 . The last 2 terms are all ionisation processes, and here again the first one will dominate at high density because it scales as n^2 , while the other scales only as n . The high density regime will be reached if the electron density exceed a critical density. The most optimistic case is to consider no radiation, which leads to

$$n_{\text{crit}} = \frac{\alpha_{\text{rec}}(T)}{\delta_{\text{rec}}(T)} \quad (3.389)$$

Using the numerical values we have computed in the previous sections for these rates, we find the critical electron density

$$n_{\text{crit}} \simeq 5.05 \times 10^{13} T^{1/2} \text{ cm}^{-3} \quad (3.390)$$

The high density regime with $n_e > n_{\text{crit}}$ will ensure most of the time that conditions are close to LTE, and one can then use the Saha relation. This regime is reached in dense stellar interiors. The low density regime with $n_e < n_{\text{crit}}$ is for everything else. It is called *coronal equilibrium* because it applies to stellar corona. This regime is of course valid in the interstellar medium and even more in the intergalactic medium. In the very low density regime, ionisation equilibrium may not be reached. This means one has to solve the full time-dependant ionisation rate equation.

Chapter 4

Collisionless self-gravitating fluids

In astrophysics, gravity (more precisely self-gravity), plays a central role. It is probably the main difference between astrophysics and many other disciplines in physics. In Chapter 1 (kinetic theory), we have introduced gravity as an external force. In Chapter 2 (astrophysical fluid dynamics), we have introduced the self-gravity of the gas also as an external force, without really justifying how we could do it. In this Chapter, we will focus on self-gravitating systems, such as galaxies or dark matter halos, for which the basic constituents, either stars or dark matter particles, can be considered as individual microscopic particles interacting through their mutual gravity. In this Chapter, gravity is not an external force anymore, it is the interaction between the particles defining the system. It plays the same role as the Coulomb interaction or the hard sphere interaction we have considered in the kinetic theory Chapter, who are considered as short-range interactions only. In self-gravitating systems, gravity plays a role both at small scale, causing collisions, and at large scale, defining what will be defined later as the coarse-grained gravitational potential.

Indeed, in this Chapter, starting from the very general "N-body system" approach, we will be able to define precisely what is a collisionless system of particles, and justify why self-gravity can be treated as an external force in the *collisionless* Boltzmann equation, although it is obviously an internal force. Very famous collisionless systems are galaxies, like our own Milky Way, made of a very large $N = 10^{12}$ ensemble of individual stars. Galaxies are also surrounded by a halo made of a very mysterious dark matter fluid, with a new, yet to be detected, type of microscopic particles that interact only through gravity. How many dark matter particles do we have in the Milky Way halo? We don't know because we don't know the dark matter particle mass. It could be $N = 10^{70}$ if the particle mass is close to the proton mass, or it could be 2000 more if it is closer to the electron mass. Other systems cannot be considered as collisionless, because they are dominated by small-range gravitational interactions. For example, globular clusters, made of $N = 10^6$ stars, are affected by collisions between stars, leading to a process called gravitational evaporation. In general, star clusters, forming out of collapsing molecular clouds, suffer from many collisional effects and as such cannot be described by the collisionless Boltzmann equation. We will ignore the theory of such collisional systems in this course, focusing only on vanilla collisionless dynamics, and describing only at the very end of the Chapter a few collisional processes, such as dynamical friction. We will solve the collisionless Boltzmann equation and find many interesting equilibrium solution in spherical and cylindrical geometry. Like for fluid dynamics, we will also study the stability of these equilibrium solution, looking for possibly stable or unstable waves.

4.1 Theory of N-body systems

We consider a system of N stars or dark matter particles with identical mass m and subject to their mutual gravitational interaction. We will follow here the formalism of Hamiltonian dynamics. Since our particles are non-relativistic and since gravity follows Newton's law, we can write the Hamiltonian of our system as:

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} - Gm^2 \sum_{i=1}^N \sum_{j=1}^{i-1} \frac{1}{|\mathbf{q}_i - \mathbf{q}_j|} \quad (4.1)$$

Note that the potential term accounts for all binary interaction, without double-counting. The system is described by the momenta and positions of each of the N particles, defining a single point in a $6N$ dimensional phase-space $(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, \dots, \mathbf{q}_N, \mathbf{p}_N)$. Following Hamiltonian's dynamics, the trajectory of each of these N particles is described by

$$\dot{\mathbf{q}}_i = \mathbf{v}_i = \frac{\partial H}{\partial \mathbf{p}_i} = \frac{\mathbf{p}_i}{m} \quad (4.2)$$

and

$$\dot{\mathbf{p}}_i = m\dot{\mathbf{v}}_i = -\frac{\partial H}{\partial \mathbf{q}_i} = -Gm^2 \sum_{j=1}^N \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} \quad (4.3)$$

Note that the mutual interaction could be in principle any type of interaction that depends only on the relative distance between particles, like the Coulomb interaction or the hard sphere interaction. We will keep this in mind later but for the time being we consider only self-gravity. Although such a dynamical Hamiltonian system conserves strictly mass and energy (and angular momentum as we will see later), it won't be able to reach an equilibrium, because stars or dark matter particles will keep on moving in complex orbits. We won't be able to find a stationary solution in such a dynamical system.

4.1.1 The N -point probability distribution function

Indeed, the particle distribution function f defined in the first Chapter (kinetic theory) as the $6D$ density of particle in phase-space at position (\mathbf{q}, \mathbf{p}) , can be written in the N -body context as

$$f(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^N \delta(\mathbf{q} - \mathbf{q}_i(t), \mathbf{p} - \mathbf{p}_i(t)) \quad (4.4)$$

where δ is the Dirac delta distribution. This approach is exact and deterministic, but not very useful in practice, especially if one hopes to find equilibrium or stationary solutions. We need to adopt a probabilistic view, where individual particle trajectories, corresponding to one single point dynamically evolving in $6N$ phase-space, are replaced by the N -body probability distribution function ρ_N , so that we can define

$$dP = \rho_N(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, \dots, \mathbf{q}_N, \mathbf{p}_N, t) d^3q_1 d^3p_1 d^3q_2 d^3p_2 \dots d^3q_N d^3p_N \quad (4.5)$$

as the probability to find our system within a small volume elements around the given position in the $6N$ dimensional phase-space. Obviously, this probability distribution function is normalized such as

$$\int_{\mathbb{R}^{6N}} dP = 1. \quad (4.6)$$

It is possible to compute various moments of any microscopic quantity Q by integrating it over phase-space. It is usually referred to as the *ensemble average* of Q and defined as

$$\bar{Q} = \int_{\mathbb{R}^{6N}} Q(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, \dots, \mathbf{q}_N, \mathbf{p}_N) \rho_N d^3q_1 d^3p_1 d^3q_2 d^3p_2 \dots d^3q_N d^3p_N \quad (4.7)$$

In particular, for each 6D phase-space coordinate (\mathbf{q}, \mathbf{p}) , we can compute the expectation value of the distribution function f as

$$\bar{f}(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^N \int_{\mathbb{R}^{6N}} \delta(\mathbf{q} - \mathbf{q}_i, \mathbf{p} - \mathbf{p}_i) \rho_N d^3q_1 d^3p_1 d^3q_2 d^3p_2 \dots d^3q_N d^3p_N \quad (4.8)$$

In each term of the sum, we can integrate out the coordinate corresponding to each δ -function and find

$$\bar{f}(\mathbf{q}, \mathbf{p}, t) = N \int_{\mathbb{R}^{6(N-1)}} \rho_N(\mathbf{q}, \mathbf{p}, \mathbf{q}_2, \mathbf{p}_2, \dots, \mathbf{q}_N, \mathbf{p}_N, t) d^3q_2 d^3p_2 \dots d^3q_N d^3p_N \quad (4.9)$$

where we used the fact that the N -point probability distribution function remains unchanged when permuting the particles. We recognize in this last equation the one-point probability distribution function defined as

$$\rho_1(\mathbf{q}_1, \mathbf{p}_1, t) = \int_{\mathbb{R}^{6(N-1)}} \rho_N(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, \dots, \mathbf{q}_N, \mathbf{p}_N, t) d^3q_2 d^3p_2 \dots d^3q_N d^3p_N \quad (4.10)$$

which is related to the particle distribution function f by

$$\bar{f}(\mathbf{q}, \mathbf{p}, t) = N \rho_1(\mathbf{q}, \mathbf{p}, t) \quad (4.11)$$

We see that in order to find an evolution equation for \bar{f} , we need to find one for ρ_1 , which means finding at least the time evolution of ρ_N .

4.1.2 Liouville's theorem

We will now derive an evolution equation for the N -point probability distribution function ρ_N using a new form of Liouville theorem. We consider an infinitesimal volume element around a given point in $6N$ -dimensional phase-space. Using a small time step Δt , we know the new position of our system in phase-space will be given by a first-order Taylor expansion of Hamilton's equations as

$$\mathbf{q}_i = \mathbf{q}_{i,0} + \frac{\partial H}{\partial \mathbf{p}_i} \Delta t \quad (4.12)$$

$$\mathbf{p}_i = \mathbf{p}_{i,0} - \frac{\partial H}{\partial \mathbf{q}_i} \Delta t \quad (4.13)$$

Note that these N different equations with $i = 1, N$ define a coordinate mapping between the two phase-spaces at $t = 0$ and at $t = \Delta t$. We would like to compute the new volume element in $6N$ -dimensional phase-space

$$dV = d^3q_1 d^3p_1 d^3q_2 d^3p_2 \dots d^3q_N d^3p_N \quad (4.14)$$

as a function of the old one

$$dV_0 = d^3q_{1,0} d^3p_{1,0} d^3q_{2,0} d^3p_{2,0} \dots d^3q_{N,0} d^3p_{N,0} \quad (4.15)$$

This is done as usual, using the determinant of the Jacobian matrix, which writes now (in 1D instead of 3D for sake of simplicity) as

$$|\det \mathbb{J}| = \prod_{i=1}^N \left| \begin{array}{cc} 1 + \frac{\partial^2 H}{\partial p_i \partial q_i} \Delta t & \frac{\partial^2 H}{\partial p_i^2} \Delta t \\ -\frac{\partial^2 H}{\partial q_i^2} \Delta t & 1 - \frac{\partial^2 H}{\partial q_i \partial p_i} \Delta t \end{array} \right| \quad (4.16)$$

For the same reasons as before, the linear terms in Δt vanish and we find that the time derivative of the volume element is zero, hence the volume element is conserved. Because the number of possible states for our system is also conserved during this Hamiltonian evolution, the probability dP is conserved. We conclude that the probability distribution function is constant in time. In other words, we have

$$\frac{d}{dt} [\rho_N(\mathbf{q}_1(t), \mathbf{p}_1(t), \mathbf{q}_2(t), \mathbf{p}_2(t), \dots, \mathbf{q}_N(t), \mathbf{p}_N(t))] = 0 \quad (4.17)$$

or, applying the chain rule, we find

$$\frac{\partial \rho_N}{\partial t} + \sum_{i=1}^N \left(\frac{\partial \rho_N}{\partial \mathbf{q}_i} \cdot \dot{\mathbf{q}}_i + \frac{\partial \rho_N}{\partial \mathbf{p}_i} \cdot \dot{\mathbf{p}}_i \right) = 0 \quad (4.18)$$

Using Hamilton's equations, we transform this into

$$\frac{\partial \rho_N}{\partial t} + \sum_{i=1}^N \left(\frac{\partial \rho_N}{\partial \mathbf{q}_i} \cdot \frac{\partial H}{\partial \mathbf{p}_i} - \frac{\partial \rho_N}{\partial \mathbf{p}_i} \cdot \frac{\partial H}{\partial \mathbf{q}_i} \right) = 0 \quad (4.19)$$

which can be written in the compact form

$$\frac{\partial \rho_N}{\partial t} + \{\rho_N, H\} = 0 \quad (4.20)$$

where we introduce the N -dimensional Poisson bracket as

$$\{A, B\} = \sum_{i=1}^N \left(\frac{\partial A}{\partial \mathbf{q}_i} \cdot \frac{\partial B}{\partial \mathbf{p}_i} - \frac{\partial A}{\partial \mathbf{p}_i} \cdot \frac{\partial B}{\partial \mathbf{q}_i} \right) \quad (4.21)$$

We see now that a stationary solution of Liouville's equation can be found requiring that $\frac{\partial \rho_N}{\partial t} = 0$. A rather general class of possible solution can be found easily with probability distribution functions of the form $\rho_N(H)$. In this case, one can indeed write the Poisson bracket as

$$\{\rho_N, H\} = \rho'_N(H) \sum_{i=1}^N \left(\frac{\partial H}{\partial \mathbf{q}_i} \cdot \frac{\partial H}{\partial \mathbf{p}_i} - \frac{\partial H}{\partial \mathbf{p}_i} \cdot \frac{\partial H}{\partial \mathbf{q}_i} \right) = 0 \quad (4.22)$$

We will use this nice property later.

4.2 The BBGKY hierarchy

In order to describe equilibrium collisionless systems, we are mostly interested in the particle distribution function (noted PDF from now on), more precisely its expectation value \bar{f} , which is proportional to the one-point probability distribution function ρ_1 . We will now derive a famous result derived independently by multiple authors across the world in the late 30's and early 40's and called the BBGKY hierarchy of equations (after Bogoliubov, Born, Green, Kirkwood and

Yvon in alphabetical order). The idea of the BBGKY equations is to connect the P -point PDF to the $(P + 1)$ -point PDF. This way, we can compute the one-point PDF ρ_1 as a function of the two-point PDF ρ_2 , and so on, until we reach the relation between ρ_{N-1} and ρ_N at the very bottom of the hierarchy.

The calculation can be performed by splitting the N -body Hamiltonian into 3 different terms $H = H_P + H_{N-P} + H_{\text{cross}}$, where the first term stands for the kinetic energy of the P first particles and the interactions between only these P first particles.

$$H_P = \sum_{i=1}^P \frac{p_i^2}{2m} - Gm^2 \sum_{i=1}^P \sum_{j=1}^{i-1} \frac{1}{|\mathbf{q}_i - \mathbf{q}_j|} \quad (4.23)$$

The second term contains the kinetic energy of the last $N - P$ particles and the interactions term of only these $N - P$ last particles.

$$H_{N-P} = \sum_{i=P+1}^N \frac{p_i^2}{2m} - Gm^2 \sum_{i=(P+1)}^N \sum_{j=(P+1)}^{i-1} \frac{1}{|\mathbf{q}_i - \mathbf{q}_j|} \quad (4.24)$$

Finally, the cross term contains no kinetic energy contribution and the cross interaction terms between the two groups of particles.

$$H_{\text{cross}} = -Gm^2 \sum_{i=(P+1)}^N \sum_{j=1}^P \frac{1}{|\mathbf{q}_i - \mathbf{q}_j|} \quad (4.25)$$

We now write the definition of the P -point PDF as

$$\rho_P(\mathbf{q}_1, \mathbf{p}_1, \dots, \mathbf{q}_P, \mathbf{p}_P, t) = \int_{\mathbb{R}^{6(N-P)}} \rho_N(\mathbf{q}_1, \mathbf{p}_1, \dots, \mathbf{q}_N, \mathbf{p}_N, t) d^3q_{(P+1)} d^3p_{(P+1)} \dots d^3q_N d^3p_N \quad (4.26)$$

We take the partial time derivative, which writes as

$$\frac{\partial \rho_P}{\partial t} = \int_{\mathbb{R}^{6(N-P)}} \frac{\partial \rho_N}{\partial t} d^3q_{(P+1)} d^3p_{(P+1)} \dots d^3q_N d^3p_N \quad (4.27)$$

Injecting Liouville's equation, we get

$$\frac{\partial \rho_P}{\partial t} = - \int_{\mathbb{R}^{6(N-P)}} \{\rho_N, H\} d^3q_{(P+1)} d^3p_{(P+1)} \dots d^3q_N d^3p_N \quad (4.28)$$

We then split the Hamiltonian into the previously discussed 3 terms, which translates into 3 different contributions to the partial time derivative of ρ_P . The first term, using notations that now should make sense, is

$$\left(\frac{\partial \rho_P}{\partial t} \right)_P = - \int_{\mathbb{R}^{6(N-P)}} \{\rho_N, H_P\} d^3q_{(P+1)} d^3p_{(P+1)} \dots d^3q_N d^3p_N \quad (4.29)$$

Since the P first particles only Hamiltonian does not depend on any of the dummy variables in the integral (they correspond to the $N - P$ last particles only), we can take it out of the integral and obtain

$$\left(\frac{\partial \rho_P}{\partial t} \right)_P = - \left\{ \int_{\mathbb{R}^{6(N-P)}} \rho_N d^3q_{(P+1)} d^3p_{(P+1)} \dots d^3q_N d^3p_N, H_n \right\} \quad (4.30)$$

In the left-hand side of the Poisson bracket, we recognise the definition of the P -point PDF, so finally this first term writes

$$\boxed{\left(\frac{\partial \rho_P}{\partial t}\right)_P} = -\{\rho_P, H_P\} \quad (4.31)$$

The second term corresponds to the Hamiltonian of the last $N - P$ particles and writes as such as

$$\left(\frac{\partial \rho_P}{\partial t}\right)_{N-P} = - \int_{\mathbb{R}^{6(N-P)}} \{\rho_N, H_{N-P}\} d^3 q_{(P+1)} d^3 p_{(P+1)} \dots d^3 q_N d^3 p_N \quad (4.32)$$

Writing the definition of the Poisson bracket using now index k (because index i and j are already taken for the Hamiltonian H_{N-P})

$$\{\rho_N, H_{N-P}\} = \sum_{k=1}^N \left(\frac{\partial \rho_N}{\partial \mathbf{q}_k} \cdot \frac{\partial H_{N-P}}{\partial \mathbf{p}_k} - \frac{\partial \rho_N}{\partial \mathbf{p}_k} \cdot \frac{\partial H_{N-P}}{\partial \mathbf{q}_k} \right) \quad (4.33)$$

we notice that the sum can actually start at index $(P + 1)$ because H_{N-P} depends only on the last $N - P$ positions and momenta. Injecting the definition of H_{N-P} , we obtain

$$\{\rho_N, H_{N-P}\} = \sum_{k=(P+1)}^N \left(\frac{\partial \rho_N}{\partial \mathbf{q}_k} \cdot \frac{\mathbf{p}_k}{m} - \frac{\partial \rho_N}{\partial \mathbf{p}_k} \cdot Gm^2 \sum_{j=(P+1)}^N \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} \right) \quad (4.34)$$

From this expression, we conclude that the second term is identically zero, because, on the left-hand side and on the right-hand side of the Poisson bracket, the two terms can be integrated by separating the positions and momenta variables. This leads to a cancellation of the integral, because the left term can be integrated formally as

$$\int_{\mathbb{R}^3} \frac{\partial \rho_N}{\partial \mathbf{q}_k} d^3 q_k = [\rho_N]_{-\infty}^{+\infty} = 0 \quad (4.35)$$

and the right term can be integrated as

$$\int_{\mathbb{R}^3} \frac{\partial \rho_N}{\partial \mathbf{p}_k} d^3 p_k = [\rho_N]_{-\infty}^{+\infty} = 0 \quad (4.36)$$

In conclusion, we have

$$\boxed{\left(\frac{\partial \rho_P}{\partial t}\right)_{N-P}} = 0 \quad (4.37)$$

Let's now deal with the third term that contains the cross interaction Hamiltonian. We have

$$\left(\frac{\partial \rho_P}{\partial t}\right)_{\text{cross}} = - \int_{\mathbb{R}^{6(N-P)}} \{\rho_N, H_{\text{cross}}\} d^3 q_{(P+1)} d^3 p_{(P+1)} \dots d^3 q_N d^3 p_N \quad (4.38)$$

Here again we have to develop the Poisson bracket using index k

$$\{\rho_N, H_{\text{cross}}\} = \sum_{k=1}^N \left(\frac{\partial \rho_N}{\partial \mathbf{q}_k} \cdot \frac{\partial H_{\text{cross}}}{\partial \mathbf{p}_k} - \frac{\partial \rho_N}{\partial \mathbf{p}_k} \cdot \frac{\partial H_{\text{cross}}}{\partial \mathbf{q}_k} \right) = - \sum_{k=1}^N \frac{\partial \rho_N}{\partial \mathbf{p}_k} \cdot \frac{\partial H_{\text{cross}}}{\partial \mathbf{q}_k} \quad (4.39)$$

where the last equality is obtained by noticing that the cross Hamiltonian does not contain any momenta-dependent terms. We now compute the gradient of the cross Hamiltonian as

$$\{\rho_N, H_{\text{cross}}\} = -Gm^2 \sum_{k=1}^P \left(\frac{\partial \rho_N}{\partial \mathbf{p}_k} \cdot \sum_{j=(P+1)}^N \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} \right) - Gm^2 \sum_{k=(P+1)}^N \left(\frac{\partial \rho_N}{\partial \mathbf{p}_k} \cdot \sum_{j=1}^P \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} \right) \quad (4.40)$$

For the same reason as above, the second sum vanishes when integrated over the $N - P$ last phase-space dummy variables. We are left with only the first term on the left that we develop and re-write switching the sum over k and the sum over j as

$$\{\rho_N, H_{\text{cross}}\} = -Gm^2 \sum_{j=(P+1)}^N \sum_{k=1}^P \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3} \cdot \frac{\partial \rho_N}{\partial \mathbf{p}_k} \quad (4.41)$$

When we perform the integral over the last $N - P$ dummy variables, we can also perform a permutation of each index j leaving the integral unchanged. This leads us to the almost final result (where we set $j = (P + 1)$)

$$\left(\frac{\partial \rho_P}{\partial t} \right)_{\text{cross}} = (N - P)Gm^2 \int_{\mathbb{R}^{6(N-P)}} \sum_{k=1}^P \frac{\mathbf{q}_k - \mathbf{q}_{(P+1)}}{|\mathbf{q}_k - \mathbf{q}_{(P+1)}|^3} \cdot \frac{\partial \rho_N}{\partial \mathbf{p}_k} d^3 q_{(P+1)} d^3 p_{(P+1)} \dots d^3 q_N d^3 p_N \quad (4.42)$$

Introducing now the $(P + 1)$ -point PDF

$$\rho_{(P+1)} = \int_{\mathbb{R}^{6(N-P-1)}} \rho_N d^3 q_{(P+2)} d^3 p_{(P+2)} \dots d^3 q_N d^3 p_N \quad (4.43)$$

we find the expression for the third term

$$\boxed{\left(\frac{\partial \rho_P}{\partial t} \right)_{\text{cross}} = (N - P)Gm^2 \int_{\mathbb{R}^6} \sum_{k=1}^P \frac{\mathbf{q}_k - \mathbf{q}_{(P+1)}}{|\mathbf{q}_k - \mathbf{q}_{(P+1)}|^3} \cdot \frac{\partial \rho_{(P+1)}}{\partial \mathbf{p}_k} d^3 q_{(P+1)} d^3 p_{(P+1)}} \quad (4.44)$$

Combining this last equation with the first term we derived (remember that the second term vanished), we finally obtain the main equation of the BBGKY hierarchy

$$\boxed{\frac{\partial \rho_P}{\partial t} + \{\rho_P, H_P\} = (N - P)Gm^2 \sum_{k=1}^P \int_{\mathbb{R}^6} \frac{\mathbf{q}_k - \mathbf{q}_{(P+1)}}{|\mathbf{q}_k - \mathbf{q}_{(P+1)}|^3} \cdot \frac{\partial \rho_{(P+1)}}{\partial \mathbf{p}_k} d^3 q_{(P+1)} d^3 p_{(P+1)}} \quad (4.45)$$

This equation can be interpreted as follows: in absence of interaction force with the $N - P$ other particles, the time evolution of the P -point PDF is also following the Liouville's equation but only for the sub-system of P particles, totally ignoring the other $N - P$ particles. The term on the right-hand side can be considered as a *collision integral* between the P particle group we have chosen to follow here and any other particle in the other $N - P$ particle group. We see that this equation sets a hierarchy in which $\frac{\partial \rho_1}{\partial t}$ depends on ρ_2 , $\frac{\partial \rho_2}{\partial t}$ depends on ρ_3 , and so on, until we reach $\frac{\partial \rho_N}{\partial t}$ given by Liouville's equation without a collision term.

4.2.1 The collisionless Boltzmann equation

As we said earlier, what we really need here to describe possible equilibrium systems is the particle distribution function \bar{f} or equivalently the one-point PDF ρ_1 , since we have the relation

$$\bar{f} = N\rho_1 \quad (4.46)$$

For $P = 1$, we have for the one-particle Hamiltonian

$$H_1 = \frac{p_1^2}{2m} \quad (4.47)$$

because a single particle does not interact gravitationally with itself. The first equation in the BBGKY hierarchy then writes

$$\frac{\partial \rho_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} = (N-1)Gm^2 \int_{\mathbb{R}^6} \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \cdot \frac{\partial \rho_2}{\partial \mathbf{p}_1} d^3 q_2 d^3 p_2 \quad (4.48)$$

We see that it depends on the two-particle PDF ρ_2 , for which the two-particle Hamiltonian writes now ($P = 2$)

$$H_2 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - Gm^2 \frac{1}{|\mathbf{q}_1 - \mathbf{q}_2|} \quad (4.49)$$

so that the second equation in the BBGKY hierarchy writes

$$\begin{aligned} \frac{\partial \rho_2}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}_1} + \frac{\mathbf{p}_2}{m} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}_2} - Gm^2 \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \cdot \left(\frac{\partial \rho_2}{\partial \mathbf{p}_1} - \frac{\partial \rho_2}{\partial \mathbf{p}_2} \right) = \\ (N-2)Gm^2 \int_{\mathbb{R}^6} \left(\frac{\mathbf{q}_1 - \mathbf{q}_3}{|\mathbf{q}_1 - \mathbf{q}_3|^3} \cdot \frac{\partial \rho_3}{\partial \mathbf{p}_1} + \frac{\mathbf{q}_2 - \mathbf{q}_3}{|\mathbf{q}_2 - \mathbf{q}_3|^3} \cdot \frac{\partial \rho_3}{\partial \mathbf{p}_2} \right) d^3 q_3 d^3 p_3 \end{aligned} \quad (4.50)$$

Be reassured, we are going to stop the hierarchy here with this second equation. It is indeed quite clear that this second equation features on the left hand side the relative force between the two interacting particles 1 and 2, while the right-hand side contains the collision integral with all possible third particles. These two contributions are fundamentally different. The left-hand side interaction is local and describes two-body interactions. One might even call these interactions ‘‘collisions’’. The right-hand side is non-local as it features an integral over the entire phase-space of particles of type 3. This is the gravitational interaction due to the large-scale distribution of matter, in which distant particles dominates the integral. Note that the first equation in the hierarchy only contains a large-scale contribution. This is why we need the second equation of the hierarchy to define precisely what is a collisionless fluid.

Because the collision integral on the right-hand side is dominated by distant particles, we can assume that the three-point PDF evaluated at the third particle coordinates is completely uncorrelated with the two collision partners. Mathematically, this writes

$$\rho_3(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, \mathbf{q}_3, \mathbf{p}_3) \simeq \rho_2(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2) \rho_1(\mathbf{q}_3, \mathbf{p}_3) \quad (4.51)$$

Note that this is equivalent to ignoring three-body collisions in setting up the properties of the system. This is usually a valid assumption for rarefied gases. The collision integral can be simplified greatly. We define first the *fluid* mass density as

$$\rho(\mathbf{q}_3) = \int_{\mathbb{R}^3} m \bar{f}(\mathbf{q}_3, \mathbf{p}_3) d^3 p_3 = Nm \int_{\mathbb{R}^3} \rho_1(\mathbf{q}_3, \mathbf{p}_3) d^3 p_3 \quad (4.52)$$

which is exactly the same definition we use in Chapter 1 on kinetic theory. We then define the total gravitational acceleration due to the entire particle distribution, also called the contribution from the *mean field* ϕ .

$$\mathbf{g}(\mathbf{q}_1) = -G \int_{\mathbb{R}^3} \frac{\mathbf{q}_1 - \mathbf{q}_3}{|\mathbf{q}_1 - \mathbf{q}_3|^3} \rho(\mathbf{q}_3) d^3 q_3 = -\frac{\partial \phi}{\partial \mathbf{q}_1} \quad (4.53)$$

where the gravitational field is given by

$$\phi(\mathbf{q}_1) = -G \int_{\mathbb{R}^3} \frac{\rho(\mathbf{q}_3)}{|\mathbf{q}_1 - \mathbf{q}_3|} d^3 q_3 \quad (4.54)$$

Note that this is the gravity we are familiar with in fluid dynamics, defined by

$$\mathbf{g} = -\nabla \phi \quad \text{with} \quad \Delta \phi = 4\pi G \rho \quad (4.55)$$

We can now simplify the collision integral quite drastically and we obtain a new right-hand side for the second equation in the hierarchy

$$\frac{N-2}{N} \left(-m\mathbf{g}(\mathbf{q}_1) \cdot \frac{\partial \rho_2}{\partial \mathbf{p}_1} - m\mathbf{g}(\mathbf{q}_2) \cdot \frac{\partial \rho_2}{\partial \mathbf{p}_2} \right) \quad (4.56)$$

If N is large enough, we can safely assume that $N-2 \simeq N$ and the second equation in the hierarchy writes finally

$$\boxed{\frac{\partial \rho_2}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}_1} + \frac{\mathbf{p}_2}{m} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}_2} - Gm^2 \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \cdot \left(\frac{\partial \rho_2}{\partial \mathbf{p}_1} - \frac{\partial \rho_2}{\partial \mathbf{p}_2} \right) \simeq -m\mathbf{g}_1 \cdot \frac{\partial \rho_2}{\partial \mathbf{p}_1} - m\mathbf{g}_2 \cdot \frac{\partial \rho_2}{\partial \mathbf{p}_2}} \quad (4.57)$$

We can now rigorously define two regimes:

- **the collisionless limit**, for which the two-particle interaction term is *always* negligible compared to the mean field contributions
- **the collisional limit**, for which the two-body interaction is *always* stronger than the mean field contribution.

Mathematically, this boils down to comparing the magnitude of these two terms

$$-Gm \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \quad \text{and} \quad \{\mathbf{g}_1, \mathbf{g}_2\} \quad (4.58)$$

In this section, we are interested in the collisionless case, so we can write the second equation of the hierarchy as this very simple equation

$$\boxed{\frac{\partial \rho_2}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}_1} + \frac{\mathbf{p}_2}{m} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}_2} + m\mathbf{g}_1 \cdot \frac{\partial \rho_2}{\partial \mathbf{p}_1} + m\mathbf{g}_2 \cdot \frac{\partial \rho_2}{\partial \mathbf{p}_2} = 0} \quad (4.59)$$

We see that the gravitational acceleration comes only from the mean field contribution, where long-range forces dominate. There is therefore no reason why strong correlations between nearby particles should appear. We can therefore make the bold statement that particles of type 1 and particles of type 2 are also uncorrelated,

$$\rho_2(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2) \simeq \rho_1(\mathbf{q}_1, \mathbf{p}_1) \rho_1(\mathbf{q}_2, \mathbf{p}_2) \quad (4.60)$$

and re-write now the first equation in the hierarchy as

$$\frac{\partial \rho_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} \simeq -\frac{N-1}{N} m\mathbf{g}_1 \cdot \frac{\partial \rho_1}{\partial \mathbf{p}_1} \quad (4.61)$$

Assuming again that N is large enough so that $N-1 \simeq N$, we finally obtain the collisionless Boltzmann equation (CBE), also known as the Vlasov equation

$$\boxed{\frac{\partial \rho_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} + m\mathbf{g}_1 \cdot \frac{\partial \rho_1}{\partial \mathbf{p}_1} = 0} \quad (4.62)$$

We see that if initially we start from an uncorrelated particle distribution with $\rho_2(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2) = \rho_1(\mathbf{q}_1, \mathbf{p}_1) \rho_1(\mathbf{q}_2, \mathbf{p}_2)$, then the second equation of the hierarchy is automatically satisfied by the first equation of the hierarchy and the property of the particles being uncorrelated is preserved at all time. These are the conditions for having a true collisionless system: uncorrelated initial conditions and gravitational interactions dominated by the mean field at all time and everywhere. In the next section, we will derive a quantitative condition for the collisionless limit to be valid, based on the *relaxation time*.

4.2.2 The collisional Boltzmann equation

Before we move on, we can discuss briefly the other limit, namely the strong collisional limit. In this case, the second equation of the BBGKY hierarchy can be written as

$$\frac{\partial \rho_2}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}_1} + \frac{\mathbf{p}_2}{m} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}_2} - Gm^2 \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \cdot \left(\frac{\partial \rho_2}{\partial \mathbf{p}_1} - \frac{\partial \rho_2}{\partial \mathbf{p}_2} \right) = 0 \quad (4.63)$$

in which we dropped the mean field contributions. We see that this equation becomes purely local in phase-space and in fact describes a collision between two particles. In this case, the two-particle PDF is strongly correlated and the right-hand side of the first equation of the hierarchy cannot be simplified as in the collisionless case. It is more convenient here to replace \mathbf{q}_1 and $\mathbf{u}_1 = \mathbf{p}_1/m$ and \mathbf{q}_2 and $\mathbf{u}_2 = \mathbf{p}_2/m$ by the centre of mass position \mathbf{Q} and velocity \mathbf{V} and the relative position \mathbf{q} and velocity \mathbf{v} defined here as

$$\mathbf{Q} = \frac{\mathbf{q}_1 + \mathbf{q}_2}{2} \quad \text{and} \quad \mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2 \quad \text{and} \quad \mathbf{V} = \frac{\mathbf{u}_1 + \mathbf{u}_2}{2} \quad \text{and} \quad \mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2 \quad (4.64)$$

Using these new variables, we can write the second equation of the BBGKY hierarchy as

$$\frac{\partial \rho_2}{\partial t} + \mathbf{V} \cdot \frac{\partial \rho_2}{\partial \mathbf{Q}} + \mathbf{v} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}} = Gm^2 \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \cdot \left(\frac{\partial \rho_2}{\partial \mathbf{p}_1} - \frac{\partial \rho_2}{\partial \mathbf{p}_2} \right) \quad (4.65)$$

Remember that, during the collision, the centre of mass velocity is conserved and, as a consequence, the centre of mass position is just a straight line. The time derivative and the spatial derivative with respect to the centre of mass can both be neglected during the collision, as they both evolve on macroscopic time scales. We therefore obtain a balance between the convective term and the force term, both depending on the relative position during the collision, which writes as

$$\mathbf{v} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}} = Gm^2 \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \cdot \left(\frac{\partial \rho_2}{\partial \mathbf{p}_1} - \frac{\partial \rho_2}{\partial \mathbf{p}_2} \right) \quad (4.66)$$

Let us write again the first equation of the hierarchy.

$$\frac{\partial \rho_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} = (N-1)Gm^2 \int_{\mathbb{R}^6} \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \cdot \frac{\partial \rho_2}{\partial \mathbf{p}_1} d^3 q_2 d^3 p_2 \quad (4.67)$$

We now modify the right-hand side by adding a term that is equal to zero, for again the same reason as before, namely that we can integrate out the derivative of ρ_2 with respect to the second momentum.

$$\frac{\partial \rho_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} = (N-1)Gm^2 \int_{\mathbb{R}^6} \frac{\mathbf{q}_1 - \mathbf{q}_2}{|\mathbf{q}_1 - \mathbf{q}_2|^3} \cdot \left(\frac{\partial \rho_2}{\partial \mathbf{p}_1} - \frac{\partial \rho_2}{\partial \mathbf{p}_2} \right) d^3 q_2 d^3 p_2 \quad (4.68)$$

We recognise in the integral the term we obtained using the second equation of the hierarchy. We can therefore replace it using collision equilibrium condition as

$$\frac{\partial \rho_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} = (N-1) \int_{\mathbb{R}^6} \mathbf{v} \cdot \frac{\partial \rho_2}{\partial \mathbf{q}} d^3 q d^3 p_2 \quad (4.69)$$

Note that we changed variables in the integral from $(\mathbf{q}_1, \mathbf{q}_2)$ to (\mathbf{Q}, \mathbf{q}) . We now choose in the integral the x-axis aligned with the relative velocity \mathbf{v} . For the perpendicular plane, we use a polar coordinate system with variable b and ϕ , where b is the impact parameter and ϕ the collision plane angle. With $d^3 q = dq b db d\phi$, we have

$$\frac{\partial \rho_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} = (N-1) \int_{\mathbb{R}^3} \int_0^{+\infty} \int_0^{2\pi} \int_{-\infty}^{+\infty} v \frac{\partial \rho_2}{\partial q} dq b db d\phi d^3 p_2 \quad (4.70)$$

We can easily integrate ρ_2 over q and obtain

$$\frac{\partial \rho_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} = (N-1) \int_{\mathbb{R}^3} \int_0^{+\infty} \int_0^{2\pi} v (\rho_2^{+\infty} - \rho_2^{-\infty}) b db d\phi d^3 p_2 \quad (4.71)$$

Long before the collision, because of the large distances involved, the 2-point PDF at $-\infty$ can be considered as uncorrelated, so that

$$\rho_2^{-\infty} \simeq \rho_1(\mathbf{Q}, \mathbf{p}_1) \rho_1(\mathbf{Q}, \mathbf{p}_2) \quad (4.72)$$

Note that here the centre of mass coordinate is considered fixed during the collision. This is not strictly correct, as only the centre of mass velocity is conserved, but can be justified by the fact that the centre of mass coordinate evolves very slowly, on a macroscopic time scale. The 2-point PDF after the collision, on the other hand, is not uncorrelated, precisely because of the effect of the collision. We can however compute its value using the reverse collision with incoming velocities \mathbf{u}'_1 and \mathbf{u}'_2 corresponding to the outgoing velocities we have for the direct collision with incoming velocities \mathbf{u}_1 and \mathbf{u}_2 . We thus write

$$\rho_2^{+\infty}(\mathbf{Q}, \mathbf{p}_1, \mathbf{p}_2) = \rho_2^{-\infty}(\mathbf{Q}, \mathbf{p}'_1, \mathbf{p}'_2) \simeq \rho_1(\mathbf{Q}, \mathbf{p}'_1) \rho_1(\mathbf{Q}, \mathbf{p}'_2) \quad (4.73)$$

Finally, multiplying the entire equation by N and assuming again that $N-1 \simeq N$, we obtain the collisional Boltzmann equation in a form identical to the one we have found in Chapter 1.

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{q}_1} = \int_{\mathbb{R}^3} \int_0^{+\infty} \int_0^{2\pi} (f'_1 f'_2 - f_1 f_2) v b db d\phi d^3 p_2} \quad (4.74)$$

For a collisional ionised gas, we have to consider two types of interaction, the gravitational interaction and the Coulomb interaction. All the previous calculations can be performed also for the Coulomb interaction, replacing Gm^2 by e^2 (using cgs units of course). On small scales, the Coulomb interaction will dominate the gravitational one and the latter can be ignored. We then find the collision integral of the Boltzmann equation based only on the Coulomb force. On scales larger than the Debye length, the plasma develops an almost perfect charge neutrality and only the gravitational interaction matters. Remember that on large scale the two-point PDF is uncorrelated and therefore the collision integral for self-gravity simplifies into the mean field contribution. This is how one justifies why we can consider self-gravity as an external force in the Boltzmann and Euler equations for self-gravitating collisional fluids (see Chapter 2).

4.2.3 The relaxation time

We have seen in the previous paragraph that the most important condition for a collisionless system is that the mean field force is much larger than the relative force during a binary collision. In other words, it means that each particle in the system follows a strict Hamiltonian evolution with a gravitational force due to the mean field, equivalent to an external gravitational field. Because the evolution is described by a one-particle Hamiltonian, without a collision term on the right-hand side, the particle total energy (kinetic plus potential) is strictly conserved along its orbit. If on the other hand, individual binary collisions play a role, we don't expect the total energy of the particle to be conserved. The system will not be described by a one-particle Hamiltonian anymore. In this section, we will compute the *relaxation time*, as the time needed for binary collisions to dissipate the orbital energy. Obviously, if this time is long enough, then we can conclude that the system is indeed collisionless. If on the other hand, the relaxation time is comparable to the orbital time, then we can conclude that collisional effects will be strong.

We consider a system of N particles (or stars) in virial equilibrium. The total mass of the system is $M = Nm$ where m is the particle mass. We also assume that the system is spherical, with volume $V = 4\pi/3R^3$. The mean particle density is $n = N/V$. We then consider one particular particle that we want to follow along its orbit. Traditionally, this particle is called the *subject star* and is labelled s . All the other particles in the system are called the *field stars* and are labelled f . We put ourselves in the frame where the subject star is at rest and its coordinates \mathbf{x}_s define the origin of the Cartesian axis. A more rigorous derivation would have considered the center of mass of the collision instead. We then consider a collision partner as one of the field stars. As always (see Chapter 1), we model the binary collision with the x -axis aligned with the direction of the relative velocity v between the two collision partners, and the impact parameter b as the distance between the incoming trajectory of the field star and the x -axis. For sake of simplicity, we assume that the relative velocity is the same for all particle and equal to v .

We use the so-called Born approximation, also named the “small deflection angle” approximation. In this case, the trajectory of the field star is just

$$x \simeq vt \quad \text{and} \quad y \simeq b \quad (4.75)$$

where the time coordinate is defined so that $t = 0$ correspond to the collision. We can compute the vertical acceleration felt by the subject star as

$$a_y = \frac{Gm}{(x^2 + y^2)^{3/2}} y \simeq \frac{Gm}{(v^2 t^2 + b^2)^{3/2}} b \quad (4.76)$$

After the collision when $t \rightarrow +\infty$, the subject star will have a net positive velocity kick in the y direction and with magnitude

$$\Delta v = \int_{-\infty}^{+\infty} a_y(t) dt \simeq \frac{2Gm}{bv} \quad (4.77)$$

Note that the velocity change in the x direction is zero because the longitudinal accelerations sum up to zero before and after the collision. After this single collision, the subject star kinetic energy was increased by the amount

$$\Delta K = \frac{1}{2} \Delta v^2 = \frac{2G^2 m^3}{b^2 v^2} \quad (4.78)$$

We now compute the rate of change of the kinetic energy of the subject star by multiplying this single discrete kinetic energy kick by the collision rate with all field stars of relative velocity v . For this, we use the collision cylinder technique that should now be familiar, with volume $dV = 2\pi b db v dt$

$$\frac{dK}{dt} = \int_{b_{\min}}^{b_{\max}} \Delta K n 2\pi b db v = \frac{4\pi G^2 m^2}{v} n \log \frac{b_{\max}}{b_{\min}} \quad (4.79)$$

We recognize here the classical Coulomb logarithm $\log \Lambda$ that appears quite naturally for gravity too. We cannot integrate between zero to infinity, as it would diverge, We need to choose a maximum and a minimum value. For the maximum impact parameter, we choose the system size $b_{\max} = R$. For the minimum value, we choose the “90 degree deflection angle” value given by

$$b_{\min} = \frac{2Gm}{v^2} \quad (4.80)$$

The justification is that those field stars will be removed from the vicinity of the orbit of the subject stars and will not contribute to the integral.

We now use the fact that our field stars are in virial equilibrium. The virial theorem writes here $2K + V = 0$ where the kinetic energy of the system is $K = \frac{1}{2}Mv^2$ and the virial is $\frac{3}{5}GM^2/R$ for a uniform sphere. This gives

$$v^2 = \frac{3}{5} \frac{GM}{R} \quad \text{and} \quad b_{\min} = \frac{10}{3} \frac{R}{N} \quad \text{and finally} \quad \log A \simeq \log N \quad (4.81)$$

We define the relaxation time as

$$\frac{dK}{dT} = \frac{K}{t_{\text{relax}}} \quad \text{where} \quad K = \frac{1}{2}v^2 \quad (4.82)$$

We want to express it in unit of the orbital time $t_{\text{orb}} = R/v$. From our previous calculations, we get

$$\frac{1}{t_{\text{relax}}} = \frac{1}{t_{\text{orb}}} \frac{8\pi Gm^2 R}{v^4} n \log N \quad (4.83)$$

Injecting the value for v^2 we found from using the virial theorem, we get

$$\frac{1}{t_{\text{relax}}} = \frac{1}{t_{\text{orb}}} \frac{200\pi m^2 R^3}{9M^2} \frac{N}{V} \log N \quad (4.84)$$

Using now $V = 4\pi/3R^3$, we finally get the famous result

$$\boxed{t_{\text{relax}} \simeq t_{\text{orb}} \frac{1}{16} \frac{N}{\log N}} \quad (4.85)$$

Interestingly enough, in our simplified approach using a uniform virialized sphere, the relaxation time depends only on the number of particles. We can now examine a few well-known stellar systems and check whether they can be considered as collisionless system or not. The Milky Way, for example, contains approximately $N = 400$ billion stars. This gives us $t_{\text{relax}} = 10^9 t_{\text{orb}}$ for an orbital time is 200 Myr. The Milky Way is clearly a collisionless stellar system. A typical globular cluster, on the other hand, can have as low as $N = 10^5$ stars. This results in $t_{\text{relax}} = 500 t_{\text{orb}}$. We have to be careful here. Indeed, globular clusters are very old objects, as old as the universe. The orbital time for globular clusters can be estimated to be 10 Myr, giving a relaxation time of 5 Gyr, smaller than the age of the Universe which is 13 Gyr. Globular clusters are therefore not collisionless systems and they cannot be described accurately by the collisionless Boltzmann equation.

4.3 Potential-density pairs

In what follows, we assume that the conditions for having a collisionless system are fulfilled. We can then now describe the distribution of the stars or dark matter particles using the distribution function f which encodes the number of stars or dark matter particle per phase-space fluid element as

$$dN = f(\mathbf{x}, \mathbf{p}) d^3x d^3p \quad (4.86)$$

Note that compared to the previous sections, we are now abandoning the notation \bar{f} . Indeed, the distribution function f should be interpreted as the *expectation* of the number of particles in the phase-space element, not the actual number which can always fluctuate with time. As the BBGKY analysis has demonstrated, the distribution function satisfies the collisionless Boltzmann equation, thereafter noted CBE, which writes

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - m \nabla \phi \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \quad (4.87)$$

where the gravitational potential is the solution of the Poisson equation

$$\Delta\phi = 4\pi G\rho \quad (4.88)$$

and the fluid density is given by the zeroth-order moment of the distribution function as

$$\rho(\mathbf{x}, t) = \int_{\mathbb{R}^3} m f(\mathbf{x}, \mathbf{p}, t) d^3p \quad (4.89)$$

In order to solve for the time evolution of f , we need to know the initial conditions at $t = 0$, then compute ρ by integrating f in momentum space, then solve for the Poisson equation and then finally update f using the CBE. In this sequence, one key step is to solve for the Poisson equation. Indeed, if one knows ρ , one can immediately deduce ϕ , assuming we have proper boundary conditions. Reversely, if one know ϕ , then using the Laplace operator will give us immediately ρ . These two scalar fields actually defines what is called a *potential-density pair*. We explain in this section how to compute potential density pair in two useful cases, namely spherical symmetry and cylindrical symmetry. But before we do that, let us first define a useful quantity, namely the gravitational potential energy associated to the potential-density pair.

4.3.1 Gravitational potential energy

An important quantity to characterize a potential-density pair is the gravitational potential energy. It is defined as the total work required to bring the equilibrium system from infinity to its current state. For an isolated particle embedded into a fixed *external* gravitational potential ϕ , this can be computed as

$$E_{\text{pot}} = \int_{+\infty}^{\mathbf{x}} \delta W = - \int_{+\infty}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{x} = m \int_{+\infty}^{\mathbf{x}} \nabla\phi \cdot d\mathbf{x} = m\phi(\mathbf{x}) \quad (4.90)$$

In this case, we see that the gravitational potential energy is equal to the gravitational potential multiplied by the particle mass. In case of a potential-density pair $(\rho(\mathbf{x}), \phi(\mathbf{x}))$, the situation is more complicated because the system is self-gravitating. We have to take into account the fact that bringing particles one by one from infinity will change the gravitational acceleration as the entire system is assembled. Let us first consider the system in an intermediate state where only a small fraction of the total mass has been already collected from infinity. If we had one additional particle at position \mathbf{x} , the increase in gravitational energy will be

$$\delta E_{\text{pot}} = m\phi(\mathbf{x}) \quad (4.91)$$

If now we collect new particles at each position in the system, we can write the total increase in potential energy as

$$\delta E_{\text{pot}} = \int_V \delta\rho(\mathbf{x})\phi(\mathbf{x})d^3x \quad (4.92)$$

where $\delta\rho$ is the small increment of density corresponding to all these new particles brought in. Note that this will correspond to a change in the gravitational potential given by Poisson equation

$$\Delta(\delta\phi) = 4\pi G\delta\rho \quad (4.93)$$

We can inject the Laplacian operator in the integral

$$\delta E_{\text{pot}} = \frac{1}{4\pi G} \int_V \Delta(\delta\phi)\phi(\mathbf{x})d^3x, \quad (4.94)$$

and use the tricks that

$$\Delta(\delta\phi) = \nabla \cdot (\nabla(\delta\phi)) \quad \text{and} \quad \nabla \cdot (\phi\nabla(\delta\phi)) = \nabla(\delta\phi) \cdot \nabla\phi + \phi\nabla \cdot \nabla(\delta\phi) \quad (4.95)$$

the second equation on the right-hand side being the famous vector relation that we have used already multiple times. We can replace in the previous integral the Laplace operator and obtain two integrals

$$\delta E_{\text{pot}} = \frac{1}{4\pi G} \int_V \nabla \cdot (\phi\nabla(\delta\phi)) d^3x - \frac{1}{4\pi G} \int_V \nabla(\delta\phi) \cdot \nabla\phi d^3x \quad (4.96)$$

Using the divergence theorem, we can show that the first integral is exactly zero as the potential vanishes at infinity. Using the following relation,

$$\delta \left(\frac{1}{2} |\nabla\phi|^2 \right) = \nabla(\delta\phi) \cdot \nabla\phi \quad (4.97)$$

we can finally write the integral as

$$\delta E_{\text{pot}} = -\frac{1}{8\pi G} \delta \left(\int_V |\nabla\phi|^2 d^3x \right) \quad (4.98)$$

Using the definition of the gravitational acceleration $\mathbf{g} = -\nabla\phi$ and adding up all the infinitesimal contributions, we finally find for the gravitational potential energy

$$\boxed{E_{\text{pot}} = -\frac{1}{8\pi G} \int_V |\mathbf{g}|^2 d^3x} \quad (4.99)$$

Note that this result allows us to also define the gravitational potential energy *density* as

$$e_{\text{pot}} = -\frac{1}{8\pi G} |\mathbf{g}|^2 \quad (4.100)$$

This will be useful in one of the next sections. Interestingly, we see that it is always negative. It also looks mathematically very similar to the electromagnetic energy density.

4.3.2 Spherically symmetric systems

In spherically symmetric systems, both the density $\rho(r)$ and the potential $\phi(r)$ depend only on the radial coordinate r . They are linked to one another by the Poisson equation, written in spherical coordinates as

$$\Delta\phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\phi}{\partial r} \right) = 4\pi G\rho \quad (4.101)$$

We have already solved this equation in Chapter 2 for the hydrostatic equilibrium equation. We will not repeat it here but just mention in Table 4.1 a few well known spherical density-potential pairs. They will be quite useful in what follows. One of the most famous profile in astrophysics is probably the Navarro, Frenk and White (NFW) profile for dark matter haloes. It has been found to fit the results of N-body simulations quite well and is now used routinely to describe the mass distribution in galaxies and galaxy clusters. The density profile looks like

$$\rho(r) = \rho_s \frac{1}{\frac{r}{r_s} \left(1 + \frac{r}{r_s} \right)^2} \quad (4.102)$$

and the cumulative mass within radius r can be integrated analytically as

$$M(< r) = \int_0^r 4\pi r'^2 \rho(r') dr' = 4\pi \rho_s r_s^3 \left(\ln \left(1 + \frac{r}{r_s} \right) - \frac{r/r_s}{1 + r/r_s} \right) \quad (4.103)$$

Name	density	potential	acceleration
Point mass	$\rho(r) = m_0\delta(r)$	$\phi(r) = -\frac{Gm_0}{r}$	$g(r) = -\frac{Gm_0}{r^2}$
Homogeneous sphere	$\rho(r) = \rho_0$	$\phi(r) = \frac{4\pi G}{3}\rho_0\left(\frac{r^2}{2} - \frac{3r_0^2}{2}\right)$	$g(r) = -\frac{4\pi G}{3}\rho_0 r$
Plummer sphere	$\rho(r) = \frac{3}{4\pi}\frac{M_0 b^2}{(r^2 + b^2)^{5/2}}$	$\phi(r) = -\frac{GM_0}{\sqrt{r^2 + b^2}}$	$g(r) = -\frac{GM_0}{(r^2 + b^2)^{3/2}}r$
Singular isothermal	$\rho(r) = \frac{\sigma_0^2}{2\pi G}\frac{1}{r^2}$	$\phi(r) = \sigma_0^2 \ln\left(\frac{r}{r_0}\right)^2$	$g(r) = -\frac{2\sigma_0^2}{r}$

Table 4.1: Example of potential-density pairs in a few well-known spherical systems.

The gravitational potential has also a closed analytical form, which makes this profile particularly attractive for many theoretical studies.

$$\phi(r) = -4\pi G r_s^2 \frac{r_s}{r} \ln\left(1 + \frac{r}{r_s}\right) \quad (4.104)$$

Interestingly enough, when one computes the circular velocity corresponding to the NFW profile, one gets

$$v_{\text{circ}}^2 = \frac{GM(<r)}{r} = 4\pi G \rho_s r_s^2 \frac{\ln\left(1 + \frac{r}{r_s}\right) - \frac{r/r_s}{1+r/r_s}}{r/r_s} \quad (4.105)$$

The circular velocity is defined by the centrifugal force a rotating fluid would need in order to balance exactly the gravitational force. The NFW circular velocity profile is quite flat, with a maximum value $v_{\text{max}} \simeq 1.64\sqrt{G\rho_s r_s^2}$ reached at radius $r_{\text{max}} \simeq 2.16r_s$. It turns out to be a good model for many galaxies whose mass distribution is dominated by dark matter.

4.3.3 Axisymmetric systems and razor thin disks

In this section, we will spend more time describing axisymmetric systems, such as self-gravitating disks. In Chapter 2, we have already studied accretion disks around a star or a black hole. We considered then that the disk mass was much smaller than the mass of the central object, so that the disk self-gravity could be ignored. In this Chapter, we want to precisely study the disk self-gravity, so we need to derive potential-density pairs for the disk component itself. Axisymmetry means here that both the density and the potential depends only on the cylindrical coordinates r and z . The pair is described by the two functions $\rho(r, z)$ and $\phi(r, z)$, connected by the Poisson equation expressed in cylindrical coordinates

$$\Delta\phi = \frac{1}{r}\partial_r(r\partial_r\phi) + \partial_z^2\phi = 4\pi G\rho(r, z) \quad (4.106)$$

We then use the technique of the separation of the variables, writing $\rho(r, z) = f_1(r)f_2(z)$. Note that we have done this before in the accretion disk case. The vertical profile can be found solving hydrostatic equilibrium and usually gives

$$\rho(r, z) = \rho_1(r) \exp\left(-\frac{z^2}{2H^2}\right) \quad (4.107)$$

where H is the disk scale height. We can then define the disk surface density $\Sigma(x, y)$ as

$$\Sigma(x, y) = \int_{\mathbb{R}} \rho(x, y, z) dz = \rho_1(r)H \quad (4.108)$$

An interesting limit, called the *razor thin disk* limit, is obtained assuming $H \rightarrow 0$ and $\rho_1 \rightarrow +\infty$ but keeping Σ finite, so that

$$\rho(x, y, z) = \Sigma(x, y)\delta(z) \quad (4.109)$$

where $\delta(z)$ is the Dirac delta function.

In order to solve for the corresponding gravitational potential, we first consider a single planar wave in the disk midplane such as

$$\Sigma(x, y) = \widehat{\Sigma}(k_x, k_y) \exp^{i(k_x x + k_y y)} \quad (4.110)$$

For the corresponding gravitational potential, we postulate a solution of the form

$$\phi(x, y, z) = \widehat{\phi}(k_x, k_y) \exp^{i(k_x x + k_y y + k_z z)} \quad (4.111)$$

We now write Poisson equation above and below the midplane as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = (-k_x^2 - k_y^2 - k_z^2) \phi = 0 \quad (4.112)$$

because $\rho = 0$ outside the disk. The only possibility to obtain a non-vanishing potential is that $k_z^2 = -k_x^2 - k_y^2$ or $k_z = \pm ik$ where $k^2 = k_x^2 + k_y^2$. Using $\phi \rightarrow 0$ for $z \rightarrow \pm\infty$, we obtain the following form

$$\phi(x, y, z) = \widehat{\phi}(k_x, k_y) \exp^{i(k_x x + k_y y)} \exp^{-k|z|} \quad (4.113)$$

In order to find the final solution for the single planar wave, we need to somehow solve Poisson equation across the disk. The second order derivative of ϕ in x and y are indeed well behaved when $|z| \rightarrow 0$, but the z derivative of ϕ is discontinuous across the disk. As a consequence the second order derivative is not defined. The only way out of this unpleasant situation is to rely on an integral form of Poisson equation. We first compute the following line integral across the disk plane

$$\int_{-\epsilon}^{+\epsilon} \Delta \phi dz = \int_{-\epsilon}^{+\epsilon} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) dz \quad (4.114)$$

The first two terms inside the integral can be combined into

$$-k^2 \widehat{\phi}(k_x, k_y) \exp^{i(k_x x + k_y y)} \int_{-\epsilon}^{+\epsilon} \exp^{-k|z|} dz \quad (4.115)$$

where here again $k^2 = k_x^2 + k_y^2$. We see that this last expression clearly converges towards zero as $\epsilon \rightarrow 0$. The third term can be simplified easily as

$$\int_{-\epsilon}^{+\epsilon} \left(\frac{\partial^2 \phi}{\partial z^2} \right) dz = \left[\frac{\partial \phi}{\partial z} \right]_{-\epsilon}^{+\epsilon} \rightarrow -2k \widehat{\phi}(k_x, k_y) \exp^{i(k_x x + k_y y)} \quad \text{for } \epsilon \rightarrow 0 \quad (4.116)$$

We can now perform the same line integral over the right-hand side of Poisson equation

$$\int_{-\epsilon}^{+\epsilon} 4\pi G \rho dz = 4\pi G \widehat{\Sigma}(k_x, k_y) \exp^{i(k_x x + k_y y)}. \quad (4.117)$$

Comparing the two sides of Poisson equation, we finally obtain the important result

$$\boxed{\widehat{\phi}(k_x, k_y) = -\frac{2\pi G}{k} \widehat{\Sigma}(k_x, k_y)} \quad (4.118)$$

We can generalise this result for an arbitrary function $\Sigma(x, y)$ using the two dimensional Fourier transform with

$$\widehat{\Sigma}(k_x, k_y) = \int_{\mathbb{R}^2} \Sigma(x, y) \exp^{-i(k_x x + k_y y)} dx dy \quad (4.119)$$

$\Sigma(r)$	$\exp^{-\alpha r}$	$\frac{1}{(r^2+\ell^2)^{1/2}}$	$\frac{1}{(r^2+l^2)^{3/2}}$
$\widehat{\Sigma}(k)$	$\frac{k}{(k^2+\alpha^2)^{3/2}}$	$\frac{1}{k} \exp^{-k\ell}$	$\frac{1}{\ell} \exp^{-k\ell}$

Table 4.2: Famous examples of axisymmetric disk surface density profiles and their corresponding Hankel transform.

and its inverse

$$\Sigma(x, y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{\Sigma}(k_x, k_y) \exp^{i(k_x x + k_y y)} dk_x dk_y \quad (4.120)$$

We obtain the final form for the potential of the razor thin disk as

$$\phi(x, y, z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{\phi}(k_x, k_y) \exp^{i(k_x x + k_y y)} \exp^{-k|z|} dk_x dk_y \quad (4.121)$$

Interestingly, everything we did so far didn't really make use of the axisymmetry of the disk. This result is in fact valid for a razor thin plane that is not necessarily axisymmetric, or invariant in the tangential direction. We will use this general property later in the Chapter. From now on, we will exploit the rotation symmetry of our disks. We can write now the 2D Fourier transform in polar (or cylindrical coordinate) r and θ as

$$\widehat{\Sigma}(k_x, k_y) = \widehat{\Sigma}(k, 0) = \int_0^{+\infty} \int_0^{2\pi} \Sigma(r) \exp^{ikr \cos \theta} r dr d\theta \quad (4.122)$$

where $k^2 = k_x^2 + k_y^2$. We recognise the Bessel function of the first kind of order 0 as

$$J_0(kr) = \frac{1}{2\pi} \int_0^{2\pi} \exp^{ikr \cos \theta} d\theta \quad (4.123)$$

which can be used to define a 1D Fourier transform in cylindrical coordinates, called *the Hankel transform* as

$$\widehat{\Sigma}(k) = 2\pi \int_0^{+\infty} \Sigma(r) J_0(kr) r dr \quad (4.124)$$

For the inverse Fourier transform, we have a similar property with

$$\Sigma(x, y) = \Sigma(r, 0) = \frac{1}{(2\pi)^2} \int_0^{+\infty} \int_0^{2\pi} \widehat{\Sigma}(k) \exp^{ikr \cos \theta} k dk d\theta \quad (4.125)$$

which can be used to define the inverse Hankel transform as

$$\Sigma(r) = \frac{1}{2\pi} \int_0^{+\infty} \widehat{\Sigma}(k) J_0(kr) k dk \quad (4.126)$$

Note that in the previous three equations, one can divide $\widehat{\Sigma}(k)$ by 2π and get another convention for the Hankel transform, for which the factor 2π has been absorbed in $\widehat{\Sigma}(k)$. In Table 4.2, we show a few examples of useful Hankel transform pairs using precisely this convention.

Kuzmin disks

We now apply this formalism to a famous model that turns out to be fully analytical and that was derived by Kuzmin. The disk surface density is given by

$$\Sigma(r) = \frac{M}{2\pi} \frac{\ell}{(r^2 + \ell^2)^{3/2}} \quad (4.127)$$

The total mass in the disk can be readily computed as

$$M_{\text{disk}} = \int_0^{+\infty} \Sigma(r) 2\pi r dr = M\ell \int_0^{+\infty} \frac{r dr}{(r^2 + \ell^2)^{3/2}} = M\ell \left[-\frac{1}{(r^2 + \ell^2)^{1/2}} \right]_0^{+\infty} = M \quad (4.128)$$

The Hankel transform can be extracted from Table 4.2.

$$\widehat{\Sigma}(k) = \frac{M}{2\pi} \exp^{-k\ell} \quad (4.129)$$

Using Poisson equation in the Fourier domain, we get

$$\widehat{\phi}(k) = -\frac{2\pi G}{k} \widehat{\Sigma}(k) = -GM \frac{\exp^{-k\ell}}{k} \quad (4.130)$$

We know that the gravitational potential is given by

$$\phi(r, z) = \frac{1}{2\pi} \int_0^{+\infty} \widehat{\phi}(k) \exp^{-k|z|} J_0(kr) k dk \quad (4.131)$$

Injecting the solution of the Poisson equation, we get

$$\phi(r, z) = -GM \frac{1}{2\pi} \int_0^{+\infty} \frac{\exp^{-k(\ell+|z|)}}{k} J_0(kr) k dk \quad (4.132)$$

We recognize another Hankel transform from Table 4.2, so we finally get

$$\boxed{\phi(r, z) = -\frac{GM}{\left(r^2 + (\ell + |z|)^2\right)^{1/2}}} \quad (4.133)$$

This fully analytical solution is quite useful in testing models and developing numerical solutions. Very far from the disk, the potential behaves like a point mass, as it should. In the disk midplane, we can compute the disk circular velocity. This velocity corresponds to the centrifugal force balancing exactly the gravitational acceleration. We get

$$v_{\text{circ}} = \sqrt{r \frac{\partial \phi(r, 0)}{\partial r}} = \sqrt{GM \frac{r^2}{(r^2 + \ell^2)^{3/2}}} \quad (4.134)$$

It reaches a maximum at $r = \sqrt{2}\ell$ and then drops on each side, similar to the rotation curves of many real galaxies.

Exponential disks

Another interesting model, probably more realistic than the Kuzmin disk, is the razor thin exponential disk. The surface density is given by

$$\Sigma(r) = \Sigma_0 \exp^{-\alpha r} \quad (4.135)$$

The disk mass associated to this profile can be easily computed using the integration by parts

$$M_{\text{disk}} = \int_0^{+\infty} \Sigma(r) 2\pi r dr = 2\pi \Sigma_0 \left(\left[-\frac{r}{\alpha} \exp^{-\alpha r} \right]_0^{+\infty} + \frac{1}{\alpha} \int_0^{+\infty} \exp^{-\alpha r} dr \right) \quad (4.136)$$

$$M_{\text{disk}} = 2\pi \Sigma_0 \frac{1}{\alpha} \left[-\frac{1}{\alpha} \exp^{-\alpha r} \right]_0^{+\infty} = \frac{2\pi \Sigma_0}{\alpha^2} \quad (4.137)$$

Using Table 4.2, we can compute the Hankel transform as

$$\widehat{\Sigma}(k) = \Sigma_0 \frac{k}{(k^2 + \alpha^2)^{3/2}} \quad (4.138)$$

Solving Poisson equation gives us

$$\widehat{\phi}(k) = -\frac{2\pi G \Sigma_0}{(k^2 + \alpha^2)^{3/2}} \quad (4.139)$$

which finally leads to the potential

$$\boxed{\phi(r, z) = -G \Sigma_0 \int_0^{+\infty} \frac{k dk}{(k^2 + \alpha^2)^{3/2}} \exp^{-k|z|} J_0(kr)} \quad (4.140)$$

which has to be numerically integrated. Interestingly, the circular velocity of the exponential disk is slightly more peaky than the equivalent Kuzmin disk (equivalent means here similar mass and size), but quite close, especially because it also reaches a maximum value close to the typical radius of the Kuzmin disk.

Mestel disk

The last razor thin model we will consider here is the Mestel disk model given by the following surface density profile

$$\Sigma(r) = \frac{v_0^2}{2\pi G r} \quad (4.141)$$

We will see how the only parameter, namely the velocity v_0 , can be interpreted. Using the Hankel transform, we find

$$\widehat{\Sigma}(k) = 2\pi \int_0^{+\infty} \frac{v_0^2}{2\pi G r} J_0(kr) r dr = \frac{v_0^2}{Gk} \int_0^{+\infty} J_0(kr) d(kr) = \frac{v_0^2}{Gk} \quad (4.142)$$

because the Bessel function is normalised so that $\int J_0(x) dx = 1$. Solving Poisson equation, we get

$$\widehat{\phi}(k) = -\frac{2\pi G \widehat{\Sigma}(k)}{k} = -\frac{2\pi v_0^2}{k^2} \quad (4.143)$$

The final 3D potential is obtained by

$$\phi(r, z) = \frac{1}{2\pi} \int_0^{+\infty} \widehat{\phi}(k) \exp^{-k|z|} J_0(kr) k dk = -v_0^2 \int_0^{+\infty} \frac{\exp^{-k|z|}}{k^2} J_0(kr) k dk \quad (4.144)$$

Looking again at Table 4.2, we don't see the result for the corresponding inverse Hankel transform. The trick is to compute the derivative of the previous integral with respect to $|z|$.

$$\frac{\partial \phi(r, z)}{\partial |z|} = v_0^2 \int_0^{+\infty} \frac{\exp^{-k|z|}}{k} J_0(kr) k dk = \frac{v_0^2}{(r^2 + |z|^2)^{1/2}} \quad (4.145)$$

where we use the inverse Hankel transform of the middle column of Table 4.2. Integrating the previous result leads to a closed analytical form for the Mestel disk potential

$$\boxed{\phi(r, z) = v_0^2 \ln \left(\frac{\sqrt{r^2 + |z|^2} + |z|}{r_0} \right)} \quad (4.146)$$

The potential in the midplane is quite simple with

$$\phi(r, 0) = v_0^2 \ln \left(\frac{r}{r_0} \right) \quad (4.147)$$

for which we can derive the circular velocity as

$$v_{\text{circ}}^2 = r \frac{\partial \phi(r, 0)}{\partial r} = v_0^2 \quad (4.148)$$

We can now interpret v_0 as being the flat circular velocity of the disk. Interestingly, Mestel disks are the prototype example of strictly *flat rotation curves*, observed in many galaxies in our vicinity. Unfortunately, the Mestel surface density profile does not match the distribution of stars in these galaxies, who are better described by exponential profiles. The reason is that the rotation curve of nearby galaxies is due to dark matter, rather than stars from which the disk is composed. In summary, a realistic galaxy like our Milky Way can be described as the sum of a spherical dark matter halo, following the NFW profile, and a razor thin exponential disk of stars.

4.4 Stellar orbits

In Chapter 1, we have seen how collision invariants could be used to derive the equilibrium distribution function assuming detailed balance in phase-space. In this Chapter, collisions are absent, so we cannot invoke a collisional equilibrium in phase-space to derive possible equilibrium distribution function. We can however study individual particle or stellar orbits and identify possible *orbital invariants*. We will see in the next section that these orbital invariants are crucial to determine the exact form of the equilibrium distribution function. The first orbital invariant is obviously the total energy, or more precisely the one-particle Hamiltonian defined by the sum of the kinetic energy and the mean field gravitational potential.

$$H(\mathbf{x}, \mathbf{p}) = \frac{p^2}{2m} + m\phi(\mathbf{x}) \quad (4.149)$$

Indeed, its time derivative is just

$$\frac{dH}{dt} = \frac{\mathbf{p}}{m} \cdot \dot{\mathbf{p}} + m\nabla\phi \cdot \dot{\mathbf{x}} = 0 \quad (4.150)$$

because the orbit of the particle is defined by the equations of motion

$$\dot{\mathbf{x}} = \frac{\mathbf{p}}{m} = \mathbf{v} \quad \text{and} \quad \dot{\mathbf{p}} = -m\nabla\phi \quad (4.151)$$

We note this first orbital invariant

$$\mathcal{I}_1 = H(\mathbf{x}, \mathbf{p}) \quad (4.152)$$

Beyond this first orbital invariant, we will discover other useful orbital invariants in different geometries. We will first focus on stellar orbits in spherical potential-density pairs, and then move to axisymmetric systems.

4.4.1 Stellar orbits in spherical potentials

Spherical systems are characterised by a spherically symmetric potential $\phi(r)$ and, as a consequence, an acceleration always pointing towards the origin of our spherical coordinate system

$$\mathbf{g}(r) = -\frac{\partial\phi}{\partial r}\mathbf{e}_r \quad (4.153)$$

We now introduce the angular momentum of the particle defined by

$$\mathbf{L} = m\mathbf{x} \times \mathbf{v} \quad (4.154)$$

For a spherical potential, with $\mathbf{x} = r\mathbf{e}_r$ and $\dot{\mathbf{v}} = g(r)\mathbf{e}_r$, we see that

$$\frac{d\mathbf{L}}{dt} = m\dot{\mathbf{x}} \times \mathbf{v} + m\mathbf{x} \times \dot{\mathbf{v}} = 0 \quad (4.155)$$

Interestingly, we just found 3 new orbital invariants as each component of the angular momentum vector

$$\mathcal{I}_2 = L_x \quad \text{and} \quad \mathcal{I}_3 = L_y \quad \text{and} \quad \mathcal{I}_4 = L_z \quad (4.156)$$

We conclude that the orbital motions will remain in a fixed plane called the orbital plane, defined by the plane perpendicular to the constant angular momentum vector.

Homogeneous sphere

A particularly simple case for our spherical potential is the one for the homogeneous sphere. Indeed, in this case, we have

$$g_x = -\frac{4\pi G}{3}\rho_0 x \quad \text{and} \quad g_y = -\frac{4\pi G}{3}\rho_0 y \quad (4.157)$$

where we assume that the orbital plane is the (x, y) Cartesian plane. The trajectory of the particle is given by Newton's second law with

$$\ddot{x} + \frac{4\pi G}{3}\rho_0 x = 0 \quad \text{and} \quad \ddot{y} + \frac{4\pi G}{3}\rho_0 y = 0 \quad (4.158)$$

We recognise two independent harmonic oscillators with the same frequency

$$\omega_0^2 = \frac{4\pi G}{3}\rho_0 \quad (4.159)$$

Orbits are ellipses with the origin at their center.

$$x = A \cos(\omega_0 t + \psi_A) \quad \text{and} \quad y = B \cos(\omega_0 t + \psi_B) \quad (4.160)$$

Kepler orbits

For more complex spherical potential, it is convenient to use 2D polar coordinates in the orbital plane (r, θ) . We can now try and solve the equation of motion of the particle using the following Lagrangian

$$\mathcal{L} = K - U = \frac{1}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - \phi(r) \quad (4.161)$$

In this coordinate system, the momentum is defined by

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} \quad \text{and} \quad p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2 \dot{\theta} \quad (4.162)$$

The tangential component of the momentum is nothing else than the specific angular momentum of the particle. The trajectory is then obtained using the Euler-Lagrange equations

$$\dot{p}_r = \frac{\partial \mathcal{L}}{\partial r} = r \dot{\theta}^2 - \frac{\partial \phi}{\partial r} \quad \text{and} \quad \dot{p}_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad (4.163)$$

The right equation just confirms that the magnitude of the angular momentum is an orbital invariant that we note here $L = r^2 \dot{\theta}$. The left equation can be interpreted as the radial acceleration resulting from the competition between the centrifugal force and the gravitational force. We can express the centrifugal force as a function of the angular momentum as

$$\ddot{r} = r \dot{\theta}^2 - \frac{\partial \phi}{\partial r} = \frac{L^2}{r^3} - \frac{\partial \phi}{\partial r} \quad (4.164)$$

where we use the substitution

$$\dot{\theta} = \frac{L}{r^2} \quad (4.165)$$

Interestingly, this last equation can be used to replace the time coordinate by the angular coordinate along the orbit as

$$\frac{dr}{dt} = \frac{L}{r^2} \frac{dr}{d\theta} \quad \text{and} \quad \frac{d^2 r}{dt^2} = \frac{L}{r^2} \frac{d}{d\theta} \left(\frac{L}{r^2} \frac{dr}{d\theta} \right) \quad (4.166)$$

Defining the new variable $u = 1/r$, we get

$$\frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta} \quad \text{and} \quad \frac{d^2 r}{dt^2} = -L^2 u^2 \frac{d^2 u}{d\theta^2} \quad (4.167)$$

and the equation of motion becomes

$$-L^2 u^2 \frac{d^2 u}{d\theta^2} = L^2 u^3 - \frac{\partial \phi}{\partial r} \quad (4.168)$$

$$\boxed{\frac{d^2 u}{d\theta^2} + u = \frac{1}{L^2 u^2} \frac{\partial \phi}{\partial r}} \quad (4.169)$$

Using the potential of a point mass, $\frac{\partial \phi}{\partial r} = Gm_0/r^2 = Gm_0 u^2$, we find the Kepler orbits with

$$\frac{d^2 u}{d\theta^2} + u = \frac{Gm_0}{L^2} \quad (4.170)$$

whose general solutions are given by

$$u = \frac{1}{r} = A \cos(\theta + \psi_A) + \frac{Gm_0}{L^2} \quad (4.171)$$

Interestingly, we recognise the classical polar form for ellipses, but this time the origin is not at the center but at the focus point of the ellipse. Note that the homogeneous sphere and the point mass potential are the only two cases that result in *closed orbits* like ellipses. Orbits for general spherically symmetric potentials can be found similarly using polar coordinates. They don't however provide simple closed orbits but instead what is called *Rosetta orbits*.

4.4.2 Stellar orbits in disks

We now describe stellar orbits in axisymmetric potentials of the form $\phi(r, z)$. For this, we use naturally cylindrical coordinates (r, θ, z) . The Lagrangian is defined as usual by

$$\mathcal{L} = K - V = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) - \phi(r, z) \quad (4.172)$$

The corresponding velocity and momentum vectors are

$$\mathbf{v} = (\dot{r}, r\dot{\theta}, \dot{z}) \quad \text{and} \quad \mathbf{p} = (\dot{r}, r^2\dot{\theta}, \dot{z}) \quad (4.173)$$

Finally, one can write the Euler-Lagrange equations as

$$\frac{d\dot{r}}{dt} = \frac{\partial \mathcal{L}}{\partial r} = r\dot{\theta}^2 - \frac{\partial \phi}{\partial r} \quad (4.174)$$

$$\frac{d\dot{\theta}}{dt} = \frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad (4.175)$$

$$\frac{d\dot{z}}{dt} = \frac{\partial \mathcal{L}}{\partial z} = -\frac{\partial \phi}{\partial z} \quad (4.176)$$

The second equation can be interpreted as the conservation of the z -component of the angular momentum, defined as

$$L_z = r^2 \dot{\theta} = \text{constant} \quad (4.177)$$

Note that in this case, only L_z is an orbital invariant, not L_x and L_y . We can exploit this new conservation law and replace $\dot{\theta}$ in the previous equations by L_z/r^2 . This leads to the definition of a new effective potential

$$\boxed{\phi_{\text{eff}}(r, z) = \phi(r, z) + \frac{L_z^2}{2r^2}} \quad (4.178)$$

and new quite simple equations of motions

$$\ddot{r} = -\frac{\partial \phi_{\text{eff}}}{\partial r} \quad (4.179)$$

$$\ddot{z} = -\frac{\partial \phi_{\text{eff}}}{\partial z} \quad (4.180)$$

Guiding center

Our new effective potential allows us to define the location of its minimum by requiring

$$\frac{\partial \phi_{\text{eff}}}{\partial r} = 0 \quad \text{and} \quad \frac{\partial \phi_{\text{eff}}}{\partial z} = 0 \quad (4.181)$$

The symmetry of the potential with respect to the vertical coordinate (see examples in the disk section of the potential-density pairs chapter) puts the minimum of the potential in the midplane with $z_{\text{min}} = 0$. We still need to find the minimum of $\phi(r, 0)$ with respect to the radial coordinate r . This defines a particular radius called *the guiding center radius* r_G which satisfies

$$\frac{\partial \phi_{\text{eff}}}{\partial r} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial r}(r_G, 0) = \frac{L_z^2}{r_G^3} \quad (4.182)$$

We see that this potential minimum corresponds to a circular orbit in the midplane. In general, stars do not follow strict circular orbits and can wander off the midplane. The guiding center only represents a virtual star on a strict coplanar circular orbit that always sits at the minimum of the effective potential.

Nearly circular orbits

In relatively thin disks, one expect the orbits to remain close to being circular. We define a new coordinate system for the stars defined relative to the guiding center with

$$x = r - r_G \quad \text{with} \quad x \ll r_G \quad \text{and} \quad z \ll r_G \quad (4.183)$$

We Taylor expand up to second order the effective potential close to the guiding center and write

$$\phi_{\text{eff}} \simeq \phi_{\text{eff}}(r_G, 0) + x \frac{\partial \phi_{\text{eff}}}{\partial r} + z \frac{\partial \phi_{\text{eff}}}{\partial z} + \frac{x^2}{2} \frac{\partial^2 \phi_{\text{eff}}}{\partial r^2} + xz \frac{\partial^2 \phi_{\text{eff}}}{\partial r \partial z} + \frac{z^2}{2} \frac{\partial^2 \phi_{\text{eff}}}{\partial z^2} \quad (4.184)$$

Because the guiding center sits at the potential minimum, the first-order derivatives are zero. The second-order cross derivative is also zero because of the symmetry with respect to the midplane. We can finally write the effective potential as

$$\phi_{\text{eff}} \simeq \phi_{\text{eff}}(r_G, 0) + \frac{1}{2} \kappa^2 x^2 + \frac{1}{2} \nu^2 z^2 \quad (4.185)$$

where we introduce the *epicyclic frequency*

$$\kappa^2 = \frac{\partial^2 \phi_{\text{eff}}}{\partial r^2}(r_G, 0) \quad (4.186)$$

and the *vertical frequency*

$$\nu^2 = \frac{\partial^2 \phi_{\text{eff}}}{\partial z^2}(r_G, 0) \quad (4.187)$$

The orbits of the stars are described by two harmonic oscillators such as

$$\ddot{x} + \kappa^2 x = 0 \quad \text{and} \quad \ddot{z} + \nu^2 z = 0 \quad (4.188)$$

These two harmonic oscillators can be used to identify three new orbital invariants, namely the vertical angular momentum and the Hamiltonian associated to each oscillator

$$\mathcal{I}_1 = L_z \quad \text{and} \quad \mathcal{I}_2 = \frac{\dot{x}^2}{2} + \frac{\kappa^2 x^2}{2} \quad \text{and} \quad \mathcal{I}_3 = \frac{\dot{z}^2}{2} + \frac{\nu^2 z^2}{2} \quad (4.189)$$

Epicyclic frequency

The two previous harmonic oscillators describe elliptical orbits with the guiding center as the center of the ellipses. Before we describe these ellipses in more details, let us compute first the value of the epicyclic frequency. We have first to compute

$$\frac{\partial \phi_{\text{eff}}}{\partial r} = \frac{\partial \phi}{\partial r} - \frac{L_z^2}{r^3} \quad (4.190)$$

so that

$$\kappa^2 = \frac{\partial^2 \phi}{\partial r^2}(r_G, 0) + 3 \frac{L_z^2}{r_G^4} \quad (4.191)$$

Because the guiding center sits at the minimum, it satisfies

$$\frac{\partial \phi}{\partial r}(r_G, 0) = \frac{L_z^2}{r_G^3} \quad (4.192)$$

We recognise here again the circular velocity and the corresponding angular frequency Ω defined as

$$v_{\text{circ}}^2 = r \frac{\partial \phi}{\partial r} \quad \text{and} \quad v_{\text{circ}} = r \Omega_{\text{circ}} \quad (4.193)$$

Injecting these new relations in the epicyclic frequency, we finally obtain

$$\boxed{\kappa^2 = r \frac{\partial}{\partial r} (\Omega_{\text{circ}}^2) + 4\Omega_{\text{circ}}^2} \quad (4.194)$$

We can evaluate this frequency for classical potentials.

- For a point mass, we have $\phi = -Gm_0/r$ and $\Omega_{\text{circ}} \propto r^{-3/2}$. This leads to $\kappa^2 = \Omega_{\text{circ}}^2$. Both frequency are equal. As a result, stars are making a complete orbit around the guiding center at exactly the same time it takes for the guiding center to orbit around the galaxy.
- For a flat rotation curve, like Mestel disks, we have $v_{\text{circ}} = \text{constant}$ and $\Omega_{\text{circ}} \propto r^{-1}$. This leads to $\kappa^2 = 2\Omega_{\text{circ}}^2$.
- Finally, for solid body rotation, which is the case for an homogeneous sphere, one has $\Omega_{\text{circ}} = \text{constant}$ and $\kappa^2 = 4\Omega_{\text{circ}}^2$.

It is tempting to summarise our previous discussion with the following property

$$\boxed{\Omega_{\text{circ}} \leq \kappa \leq 2\Omega_{\text{circ}}} \quad (4.195)$$

The epicycle ellipse

We now solve for the solution of the stellar orbital motions around the guiding center. Since the guiding center is orbiting around the galaxy in a perfect circle, we can follow it using its angular coordinate θ_G . A star, on the other hand, we have in general a different angular coordinate θ , and, more importantly, this angular coordinate will not necessarily follow the guiding center, unless it is also on the same circular orbit. We now introduce a new Cartesian coordinate

$$y = r_G \sin(\theta - \theta_G) \simeq r_G (\theta - \theta_G) \quad (4.196)$$

which represents the perpendicular coordinate of the star in the co-rotating frame. If $\theta > \theta_G$, then the star is in advance with respect to the guiding center and $y > 0$. If $\theta < \theta_G$, then the star is lagging behind the guiding center and $y < 0$. We already now that the vertical angular momentum is an orbital invariant, so we can write for this particular star

$$L_z = v_\theta r = r^2 \dot{\theta} = \text{constant} \quad (4.197)$$

Note that the radius of the star is given by $r = r_G + x$ so that we can write

$$\dot{\theta} = \frac{L_z}{r^2} = \frac{L_z}{r_G^2} \frac{1}{(1 + x/r_G)^2} \quad (4.198)$$

Within our approximation of nearly circular orbits, we have $x \ll r_G$, so we can Taylor expand this last expression, leading to

$$\dot{\theta} \simeq \frac{L_z}{r_G^2} \left(1 - 2 \frac{x}{r_G} \right) \quad (4.199)$$

The guiding center, on the other hand, follows a strict circular orbit, for which

$$\dot{\theta}_G = \Omega_{\text{circ}} \quad (4.200)$$

We now consider all the stars that have the same vertical angular momentum as the guiding center. This means that L_z can be written for both the guiding center and all the stars associated to it as

$$L_z = \Omega_{\text{circ}} r_G^2 \quad (4.201)$$

We can finally compute the evolution of the Cartesian coordinate y as

$$\boxed{\dot{y} = r_G (\dot{\theta} - \dot{\theta}_G) = -2\Omega_{\text{circ}} x} \quad (4.202)$$

We can now solve for the elliptical orbit, writing first for the radial and vertical coordinates $x = r - r_G$ and z

$$x = X \cos(\kappa t + \psi_X) \quad \text{and} \quad z = Z \cos(\nu t + \psi_Z) \quad (4.203)$$

We deduce from the equation for \dot{y} that

$$y = -Y \sin(\kappa t + \psi_Y) \quad \text{where} \quad Y = 2X \frac{\Omega_{\text{circ}}}{\kappa} \quad (4.204)$$

Because κ is bounded by Ω and 2Ω , we see that $X \leq Y \leq 2X$. We see that stars orbit around an ellipse elongated in the tangential direction, when looking down the disk face-on. This ellipse is called the *epicycle ellipse*. Stars are rotating around the guiding center in a clock-wise manner, while the guiding center is rotating in the galaxy counter-clock-wise.

4.5 Stationary solutions of the collisionless Boltzmann equation

After first justifying the use of the CBE to describe our collisionless N-body system, we have then describe the orbits of stars in several typical potential-density pairs relevant for most galaxies. We now want to find possible stationary distribution function that satisfies the CBE on one hand, and the Poisson equation on the other hand, giving us a fully self-consistent self-gravitating equilibrium system.

4.5.1 Jeans theorem

One important result from the previous section was the existence of several orbital invariants. Let's call these various invariants $\mathcal{I}_1, \mathcal{I}_2$, etc. These orbital invariants depends *only* on the Hamiltonian variables \mathbf{x} and \mathbf{p} . Importantly, they do not depend explicitly on time. The fact that they are constant along the orbit can be written as

$$\frac{d\mathcal{I}}{dt} = \frac{\partial\mathcal{I}}{\partial\mathbf{x}}\dot{\mathbf{x}} + \frac{\partial\mathcal{I}}{\partial\mathbf{p}}\dot{\mathbf{p}} = 0 \quad (4.205)$$

Jeans theorem goes as follows: “*Distribution functions that are function of the orbital invariants satisfies the stationary CBE.*”. Indeed, the stationary CBE writes as

$$\frac{\partial f}{\partial\mathbf{x}}\dot{\mathbf{x}} + \frac{\partial f}{\partial\mathbf{p}}\dot{\mathbf{p}} = 0 \quad (4.206)$$

If we have $f(\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n)$, we can inject this form into left-hand side of the stationary CBE. Using the chain rule, we obtain

$$\left(\sum_{i=1}^n \frac{\partial f}{\partial\mathcal{I}_i} \frac{\partial\mathcal{I}_i}{\partial\mathbf{x}} \right) \dot{\mathbf{x}} + \left(\sum_{i=1}^n \frac{\partial f}{\partial\mathcal{I}_i} \frac{\partial\mathcal{I}_i}{\partial\mathbf{p}} \right) \dot{\mathbf{p}} \quad (4.207)$$

which can be re-arranged as

$$\sum_{i=1}^n \frac{\partial f}{\partial\mathcal{I}_i} \left(\frac{\partial\mathcal{I}_i}{\partial\mathbf{x}}\dot{\mathbf{x}} + \frac{\partial\mathcal{I}_i}{\partial\mathbf{p}}\dot{\mathbf{p}} \right) = 0. \quad (4.208)$$

This proves that f satisfies the stationary CBE. This theorem is quite powerful, in the sense that for a given geometry, we have already identify many different orbital invariants. We just have to postulate distributions of the form presented here and we know they will automatically satisfy the stationary CBE. Note that this approach is quite similar to what we found for the Liouville theorem applied to the N-point PDF ρ_N . We have shown previously that if the N-point PDF is of the form $\rho_N(H)$ where H is the N-particle Hamiltonian, then it was automatically satisfying the stationary Liouville equation. Jeans theorem is applied only to the distribution function, or equivalently to the one-point PDF ρ_1 , but it is valid for more than one orbital invariant.

4.5.2 Distribution functions for spherical systems

The Jeans theorem is a wonderful strategy to find stationary solutions of the CBE. We have shown in the previous section that spherical systems have 4 orbital invariants, namely $\mathcal{I}_1 = E$, $\mathcal{I}_2 = L_x$, $\mathcal{I}_3 = L_y$ and $\mathcal{I}_4 = L_z$. The simplest choice would be to adopt $f(E)$ as our Ansatz. This leads to what is called *ergodic* or *isotropic* distribution functions. The most general choice would be $f(H, L_x, L_y, L_z)$, leading to what is called *anisotropic* distributions. In practice, however, only anisotropic distributions of the form $f(E, L)$ has been proposed in the literature, the most famous

one being the Osipkov-Merritt distribution. In this section, we only study ergodic solutions $f(E)$ for which

$$E = \frac{1}{2}v^2 + \phi(r) \quad (4.209)$$

We can easily compute the different moments of f like

$$\rho(r) = \int_0^{+\infty} f(E)4\pi v^2 dv \quad \text{and} \quad \rho\bar{\mathbf{v}} = \int_{\mathbb{R}^3} f(E)\mathbf{v}d^3v = 0 \quad (4.210)$$

The second equality on the right-hand side was obtained using the fact that f is isotropic and even in velocity space, while the velocity is odd. This is why the mean (or fluid) velocity vanishes automatically. Note that we use here again the convention that the fluid quantities are defined as averages of the particle (or microscopic) quantities in velocity space. We can also compute the second order moments defined as

$$\rho\sigma_{3D}^2 = \int_0^{+\infty} f(E)v^24\pi v^2 dv = 3\rho\sigma^2 \quad (4.211)$$

where $\sigma(r)$ is the 1D velocity dispersion of the underlying particle distribution at that particular radius. It is customary at this point to change variables, using

$$\psi = -\phi \quad \text{and} \quad \epsilon = -E = \psi - \frac{1}{2}v^2 \quad (4.212)$$

These variables are called *the relative potential* for ψ and the *binding energy* for ϵ . While ψ is usually always positive, the binding energy can be used to differentiate a bound particle if $\epsilon > 0$ from an unbound particle if $\epsilon \leq 0$. In our quest for stationary distribution function, we do not allow for unbound particles to be present. Indeed, one can always wait long enough for these particles to leave the system. The distribution function will thus always satisfy

$$f(\epsilon) \geq 0 \quad \text{for} \quad \epsilon > 0 \quad \text{and} \quad f(\epsilon) = 0 \quad \text{for} \quad \epsilon \leq 0 \quad (4.213)$$

We see that the condition for a particle to be bound can be written as

$$\epsilon = \psi - \frac{1}{2}v^2 > 0 \quad \text{or} \quad v < v_{\text{escape}} = \sqrt{2\psi(r)} \quad (4.214)$$

This introduces the notion of *escape velocity* as the velocity a particle needs to be unbound and to escape the system. We can exploit this new definition and rewrite the zeroth order moment as

$$\rho(r) = \int_0^{v_{\text{escape}}} f(E)4\pi v^2 dv \quad (4.215)$$

because we now know that f is zero for higher velocities. We now switch to the binding energy variable, writing

$$v = \sqrt{2(\psi - \epsilon)} \quad (4.216)$$

We obtain

$$\boxed{\rho(\psi) = 4\pi\sqrt{2} \int_0^\psi \sqrt{\psi - \epsilon} f(\epsilon) d\epsilon} \quad (4.217)$$

We use the notation $\rho(\psi)$ because ρ now depends explicitly on ψ and implicitly on the radius. This is a very useful form, because we also know that the potential-density pair is also satisfying Poisson equation

$$\Delta\psi = -4\pi G\rho \quad (4.218)$$

We have two equations with two unknown variables, so the problem appears a priori solvable.

Eddington formula

In the previous section, we have shown that if one knows an explicit form for f , then one can deduce a relation between ρ and ψ . Using finally Poisson equation, the whole equilibrium solution can be found. In this section, we will do the opposite, namely determine f once we have adopted a given potential-density pair. Taking the derivative of the density with respect to ψ , replacing temporarily the upper bound of the integral by $+\infty$, we obtain

$$\frac{d\rho}{d\psi} = 4\pi\sqrt{2} \int_0^\psi \frac{1}{2\sqrt{\psi-\epsilon}} f(\epsilon) d\epsilon \quad (4.219)$$

that can be re-arranged as

$$\frac{1}{\pi\sqrt{8}} \frac{d\rho}{d\psi} = \int_0^\psi \frac{f(\epsilon)}{\sqrt{\psi-\epsilon}} d\epsilon \quad (4.220)$$

The expert eyes would immediately recognise this as a convolution using the Laplace transform. Let us recall that the Laplace transform of a function $f(x)$ is defined in this context as

$$\widehat{f}(s) = \int_0^{+\infty} f(x) \exp^{-sx} dx \quad (4.221)$$

The convolution of two functions is defined as

$$(f * g)(y) = \int_0^y f(x)g(y-x)dx \quad (4.222)$$

We might list a few important properties of the Laplace transform. First, the convolution theorem states that the Laplace transform of a convolution is the product of the Laplace transforms of the two functions.

$$(f * g)(x) \leftarrow \text{Laplace transform} \rightarrow \widehat{f}(s)\widehat{g}(s) \quad (4.223)$$

The derivative of a function has also a simple Laplace transform given by

$$f'(x) \leftarrow \text{Laplace transform} \rightarrow s\widehat{f}(s) - f(0) \quad (4.224)$$

Finally, the following identity will prove handy later

$$\frac{1}{\sqrt{x}} \leftarrow \text{Laplace transform} \rightarrow \sqrt{\frac{\pi}{s}} \quad (4.225)$$

Going back to the problem at hand, we define the new function g

$$g = \frac{1}{\pi\sqrt{8}} \frac{d\rho}{d\psi} \quad (4.226)$$

As we already noticed, it can be written as a convolution with

$$g(\psi) = f(\epsilon) * \frac{1}{\sqrt{\epsilon}} \quad (4.227)$$

If we perform a Laplace transform of all these functions, we get

$$\widehat{g}(s) = \widehat{f}(s) \sqrt{\frac{\pi}{s}} \quad (4.228)$$

We now invert this relation and re-arrange it as follows

$$\widehat{f}(s) = \widehat{g}(s) \sqrt{\frac{s}{\pi}} = \widehat{g}(s) \frac{s}{\pi} \sqrt{\frac{\pi}{s}} = \left(\frac{1}{\pi} \widehat{g}(s) \sqrt{\frac{\pi}{s}} \right) s \quad (4.229)$$

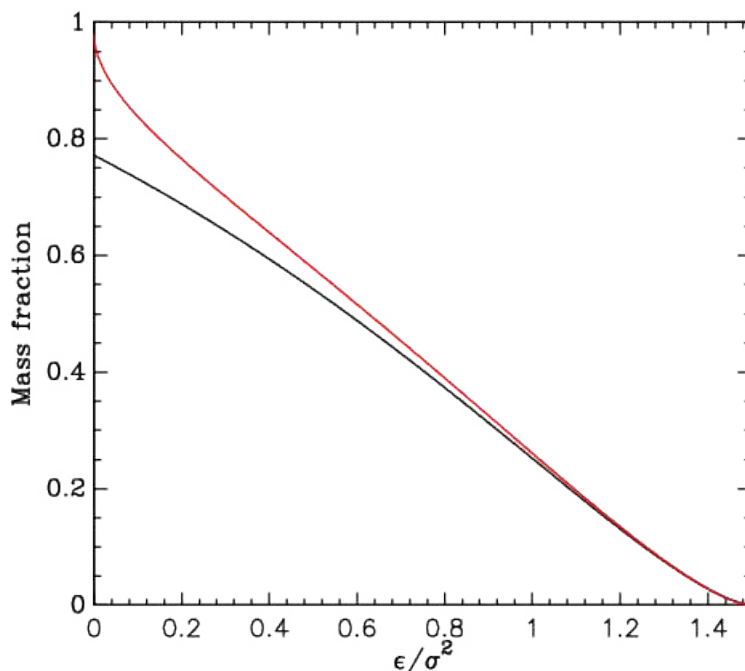


Figure 4.1: Comparison of mass distribution between the truncated Maxwell-Boltzmann distribution (black) and the true distribution (red) obtained by solving numerically the Eddington formula for the Singular Isothermal Sphere. The x-axis shows the binding energy with maximum value equal to the local relative potential at the particular radius for which $\psi = 1.5\sigma^2$.

We can use the assumption that $\rho \rightarrow 0$ fast enough, as $r \rightarrow +\infty$ and $\psi \rightarrow 0$ to conclude that $g(0) = 0$. We can finally perform the inverse Laplace transform of the last equation, recognising first again the convolution between g and $1/\sqrt{\psi}$ and then using the property of the derivative of the Laplace transform, so that

$$f(\epsilon) = \frac{d}{d\epsilon} \left(\frac{1}{\pi} \int_0^\epsilon \frac{g(\psi)}{\sqrt{\epsilon - \psi}} d\psi \right) \quad (4.230)$$

Injecting the definition of $g(\psi)$, we finally obtain the famous Eddington formula

$$f(\epsilon) = \frac{1}{\pi^2 \sqrt{8}} \frac{d}{d\epsilon} \left(\int_0^\epsilon \frac{d\rho}{d\psi} \frac{d\psi}{\sqrt{\epsilon - \psi}} \right) \quad (4.231)$$

This last equation is quite powerful, as it allows us to first choose a density profile $\rho(r)$, then solve Poisson equation to get $\psi(r)$ and finally obtain $f(\epsilon)$ directly. We could do this interesting exercise for any of the potential-density pairs we have presented in the previous sections. Most of them require unfortunately a numerical evaluation of the Eddington formula.

The Singular Isothermal Sphere and the Maxwell-Boltzmann distribution

To illustrate the power of the Eddington formula, we now study the Singular Isothermal Sphere. This is a famous potential-density pair for which the density is given by

$$\rho(r) = \rho_0 \left(\frac{r}{r_0} \right)^{-2} \quad (4.232)$$

The meaning of r_0 and its corresponding density ρ_0 will be explained later. We can now solve Poisson equation, using ψ instead of ϕ .

$$\Delta\psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) = -4\pi G\rho = -4\pi G\rho_0 r_0^2 \frac{1}{r^2} \quad (4.233)$$

We get after one integration

$$r^2 \frac{\partial\psi}{\partial r} = C_0 - 4\pi G\rho_0 r_0^2 r \quad (4.234)$$

We set the constant C_0 to zero to remove the spurious point mass solution. We then integrate a second time to get

$$\psi = \psi_0 - 4\pi G\rho_0 r_0^2 \log r = 4\pi G\rho_0 r_0^2 \log \left(\frac{r_0}{r} \right) \quad (4.235)$$

where we have chosen to set $\psi = 0$ at $r = r_0$. This is the classical problem with the SIS, namely that the potential diverges at infinity, so we have to choose a maximum radius r_0 and set the potential to zero at that finite radius. Note that the relative potential can be re-formulated by

$$\psi = 2\pi G\rho_0 r_0^2 \log \left(\frac{r_0}{r} \right)^2 = 2\pi G\rho_0 r_0^2 \log \left(\frac{\rho}{\rho_0} \right) \quad (4.236)$$

We define (quite arbitrarily for now) the velocity dispersion σ_0 as

$$\sigma_0^2 = 2\pi G\rho_0 r_0^2 \quad (4.237)$$

and finally express the potential-density pair in a compact and explicit form as

$$\boxed{\rho = \rho_0 \exp^{\psi/\sigma_0^2}} \quad (4.238)$$

We can now use this formulation in the Eddington formula, with

$$\frac{d\rho}{d\psi} = \frac{\rho_0}{\sigma_0^2} \exp^{\psi/\sigma_0^2}, \quad (4.239)$$

we obtain

$$f(\epsilon) = \frac{1}{\pi^2 \sqrt{8}} \frac{\rho_0}{\sigma_0^2} \frac{d}{d\epsilon} \left(\int_0^\epsilon \exp^{\psi/\sigma_0^2} \frac{d\psi}{\sqrt{\epsilon - \psi}} \right) \quad (4.240)$$

We change the dummy integration variable from ψ to t such that

$$t = \sqrt{2(\epsilon - \psi)} \quad \text{or} \quad \psi = \epsilon - \frac{1}{2}t^2 \quad \text{and} \quad d\psi = -tdt \quad (4.241)$$

which leads to the relatively cumbersome final form

$$\boxed{f(\epsilon) = \frac{1}{\pi^2 \sqrt{8}} \frac{\rho_0}{\sigma_0^2} \frac{d}{d\epsilon} \left(\sqrt{2} \exp^{\epsilon/\sigma_0^2} \int_0^{\sqrt{2\epsilon}} \exp^{-\frac{t^2}{2\sigma_0^2}} dt \right)} \quad (4.242)$$

We can try to simplify our result by noticing that the upper bound of the integral is the escape velocity of our stellar particles. If particles are sufficiently bound, one expect their typical binding energy to be larger than their typical kinetic energy. This translates into the condition $\epsilon \gg \sigma_0^2$, or equivalently setting the upper bound of the integral to $+\infty$. We then get

$$f(\epsilon) \simeq \frac{1}{\pi^2 \sqrt{8}} \frac{\rho_0}{\sigma_0^2} \frac{d}{d\epsilon} \left(\sqrt{\pi} \sigma_0 \exp^{\epsilon/\sigma_0^2} \right) \quad (4.243)$$

Computing the derivative with respect to ϵ , we finally obtain

$$f(\epsilon) \simeq \frac{\rho_0}{(2\pi\sigma_0^2)^{3/2}} \exp^{\epsilon/\sigma_0^2} = \frac{\rho_0}{(2\pi\sigma_0^2)^{3/2}} \exp^{\psi/\sigma_0^2} \exp^{-\frac{v^2}{2\sigma_0^2}} \quad (4.244)$$

which is the Maxwell-Boltzmann distribution.

$$\boxed{f(\epsilon) = \frac{\rho(r)}{(2\pi\sigma_0^2)^{3/2}} \exp^{-\frac{v^2}{2\sigma_0^2}}} \quad (4.245)$$

It is quite spectacular to discover the Maxwell-Boltzmann distribution in the context of collisionless fluids. Remember that we derived it in Chapter 1 for strongly collisional fluids. The reason is to be found in the notion of statistical equilibrium in phase-space. In the collisional case, this equilibrium is reached through collisions, while in the collisionless case, it is reached through different orbits filling up phase-space. Note however that the exact solution in this case is not the Maxwell-Boltzmann but the more elaborate form we derived above. Indeed, an obvious problem arises with the Maxwell-Boltzmann distribution: particle velocities are not bounded, although they should obviously be bounded by the escape velocity at each radius. A better approximation is given by the *truncated Maxwellian*

$$f(r, v) = \frac{\rho(r)}{(2\pi\sigma_0^2)^{3/2}} \exp^{-\frac{v^2}{2\sigma_0^2}} \quad \text{for } v < v_{\text{escape}}(r) \quad \text{and} \quad f(r, v) = 0 \quad \text{otherwise.} \quad (4.246)$$

We also see that σ_0 is indeed the constant velocity dispersion of the stars orbiting in the SIS. This truncated Maxwellian approach is quite useful to get a first rough approximation of the distribution function for any potential-density pair. Keep in mind that the only accurate approach would be to solve the Eddington formula.

4.5.3 Schwarzschild distribution for disks

We now move towards the problem of finding equilibrium distribution functions for axisymmetric systems like disk. We already know quite a lot on stellar orbits in disks, in particular we have identified 3 orbital invariants, namely

$$L_z = rv\theta \quad \text{and} \quad H_z = \frac{v_z^2}{2} + \frac{\nu^2 z^2}{2} \quad \text{and} \quad H_r = \frac{\dot{x}^2}{2} + \frac{\kappa^2 x^2}{2} \quad (4.247)$$

Remember that $x = r - r_G$ was the coordinate in the radial direction relative to the guiding center with radius r_G set by the vertical angular momentum $L_z = v_{\text{circ}} r_G$. In this same rotating frame, we defined also coordinate $y = r_G (\theta - \theta_G)$. We found in particular the following relation between \dot{y} and x that allowed us to derive the epicycle ellipse.

$$\dot{y} = r_G (\dot{\theta} - \dot{\theta}_G) = -2\Omega_{\text{circ}} x \quad (4.248)$$

Using $v_\theta = r\dot{\theta} \simeq r_G \dot{\theta}$ and $v_{\text{circ}} = r_G \dot{\theta}_G$, we get

$$x = \frac{v_{\text{circ}} - v_\theta}{2\Omega_{\text{circ}}} \quad (4.249)$$

We can therefore re-write the radial Hamiltonian as

$$H_r = \frac{v_r^2}{2} + \frac{\kappa^2}{4\Omega_{\text{circ}}^2} (v_\theta - v_{\text{circ}})^2 \quad (4.250)$$

We now follow Jeans theorem and introduce the following Ansatz for f

$$f(\mathbf{x}, \mathbf{v}) = S(L_z)T(H_z)U(H_r) \quad (4.251)$$

Note that compare to a strict application of Jeans theorem, we also consider here a model with proper separation of variables. We then follow the Maxwell-Boltzmann prescription derived in the previous section, writing f as

$$f(\mathbf{x}, \mathbf{v}) = S(rv_{\text{circ}}) \exp^{-H_z/\sigma_z^2} \exp^{-H_r/\sigma_r^2} \quad (4.252)$$

where σ_r and σ_r are two parameters of our model that need to be determined separately. Note that in the last equation, the radial coordinate r is identified to the guiding center radius of our nearly circular orbits. We can now inject the different equations we obtained for the two Hamiltonians

$$f(r, z, v_r, v_\theta, v_z) = S(rv_{\text{circ}}) \exp^{-\frac{v_z^2}{2\sigma_z^2}} \exp^{-\frac{v_r^2}{2\sigma_r^2}} \exp^{-\frac{v_\theta^2}{2\sigma_\theta^2}} \exp^{-\frac{(v_\theta - v_{\text{circ}})^2}{2\sigma_\theta^2}} \quad (4.253)$$

where the tangential velocity dispersion is found to be equal to

$$\sigma_\theta = \frac{2\Omega_{\text{circ}}}{\kappa} \sigma_r \quad (4.254)$$

a result that will prove useful later. We also notice that the vertical structure of the disk is also a Gaussian, like the accretion disk in vertical hydrostatic equilibrium, with disk scale height

$$H = \frac{\sigma_z}{\nu} \quad (4.255)$$

We can finally determine the function $S(r)$ by computing the zeroth-order moment

$$\rho(r, z) = \int_{\mathbb{R}^3} f d^3v = S(r) \exp^{-\frac{z^2}{2H^2}} (2\pi)^{3/2} \sigma_r \sigma_\theta \sigma_z \quad (4.256)$$

Finally, integrating vertically, we get

$$\Sigma(r) = \int_{-\infty}^{+\infty} \rho(r, z) dz = S(r) (2\pi)^2 \sigma_r \sigma_\theta \sigma_z H \quad (4.257)$$

This determines completely the distribution function, called the *Schwarzschild distribution*

$$f(r, z, v_r, v_\theta, v_z) = \frac{\Sigma(r)}{(2\pi)^2 \sigma_r \sigma_\theta \sigma_z H} \exp^{-\frac{z^2}{2H^2}} \exp^{-\frac{v_r^2}{2\sigma_r^2}} \exp^{-\frac{v_\theta^2}{2\sigma_\theta^2}} \exp^{-\frac{(v_\theta - v_{\text{circ}})^2}{2\sigma_\theta^2}} \quad (4.258)$$

Note that we derived this distribution function in the limit of nearly circular orbits. This translates into the following condition on the two main parameters

$$\sigma_r \ll v_{\text{circ}} \quad \text{and} \quad \sigma_z \ll v_{\text{circ}} \quad (4.259)$$

As a consequence, the disk is also very thin, with $H \ll r$. This explains why the mean tangential velocity of the particle is equal to the circular velocity

$$\overline{v_\theta} = v_{\text{circ}} \quad (4.260)$$

As we will see later, this is not true anymore for thicker disks. We have also to be careful with the Maxwell-Boltzmann approximation. Some of the particles will indeed escape if their kinetic energy exceeds their potential energy. An additional constraint $v < v_{\text{escape}}$ can be introduced in the model as an easy way to fix it. Note that it is also possible to derive a generalisation of the Eddington formula for axisymmetric systems, leading to very accurate equilibrium models for disks.

4.6 Jeans equation

In the previous section, we have used the Maxwell-Boltzmann approximation to find useful equilibrium distribution functions for collisionless systems. The key parameters of these distributions are the mean (or bulk) velocities and the velocity dispersions. A nice way to constrain these parameters is to compute the first moments of the CBE, exactly like in Chapter 1 when we derived the Euler equations. In the context of collisionless fluids, the equations we find are called Jeans equations. We can write again the CBE as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (4.261)$$

It is identical to the Boltzmann equation, without the collision integral on the right-hand side. The zeroth-order moment of the CBE leads to the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho \bar{v}_j) = 0 \quad (4.262)$$

The first-order moment can be obtained by multiplying the CBE by v_i and integrate over velocity space. After a derivation identical to what we did in Chapter 1, we obtain

$$\frac{\partial \rho \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho \bar{v}_i \bar{v}_j + \rho \sigma_{ij}^2) = -\rho \frac{\partial \phi}{\partial x_i} \quad (4.263)$$

where the velocity dispersion tensor is defined as

$$\rho \sigma_{ij}^2 = \int_{\mathbb{R}^3} (v_i - \bar{v}_i)(v_j - \bar{v}_j) f d^3v \quad (4.264)$$

If we look for a stationary solution, we can set all time derivatives to zero, but this leaves us with 9 unknowns with only 4 equations. We cannot solve the problem in the general case. If we add the conditions that the mean flow is static $\bar{v}_i = 0$ and that the distribution function is isotropic, we find the hydrostatic equation we have derived in Chapter 2. Unfortunately, collisionless systems are not necessarily isotropic in velocity space, except for a simple ergodic spherical system like the singular isothermal sphere.

4.6.1 Spherical systems and the anisotropy parameter

We will now derive Jeans equation for a spherical system with $\phi(r)$ and $\rho(r)$ depending only on the radial coordinate. Unfortunately, we have to consider now all the orbits of the particles defining the system, so we cannot put ourselves in the orbital plane of a single orbit anymore. We have to use the full 3D spherical coordinates (r, θ, ϕ) . In this coordinate system, the velocity of a particle is given by

$$v_r = \dot{r} \quad \text{and} \quad v_\theta = r\dot{\theta} \quad \text{and} \quad v_\phi = r \sin \theta \dot{\phi} \quad (4.265)$$

As always, we use the Lagrangian $\mathcal{L} = K - V = \frac{1}{2}v^2 - \phi(r)$. This leads to the following momentum coordinates (p_r, p_θ, p_ϕ) with

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} = v_r \quad \text{and} \quad p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2 \dot{\theta} = r v_\theta \quad \text{and} \quad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \sin^2 \theta \dot{\phi} = r \sin \theta v_\phi \quad (4.266)$$

We can also write the Euler-Lagrange equations in this coordinate system as

$$\dot{p}_r = \frac{\partial \mathcal{L}}{\partial r} = r\dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2 - \frac{\partial \phi}{\partial r} \quad (4.267)$$

$$\dot{p}_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = r^2 \sin \theta \cos \theta \dot{\phi}^2 \quad (4.268)$$

$$\dot{p}_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (4.269)$$

A striking feature of these equations is the absence of an explicit dependency with respect to ϕ . This leads us to choose for f a relatively general Ansatz of the form

$$f(r, \theta, v_r, v_\theta, v_\phi) \quad (4.270)$$

where we only removed a dependency with ϕ . The stationary collisionless Boltzmann equation can now be written in these generalised coordinates as

$$\dot{r} \frac{\partial f}{\partial r} + \dot{\theta} \frac{\partial f}{\partial \theta} + \dot{p}_r \frac{\partial f}{\partial p_r} + \dot{p}_\theta \frac{\partial f}{\partial p_\theta} = 0 \quad (4.271)$$

Replacing time derivatives with momentum variables, we finally obtain the CBE in spherical coordinates

$$\boxed{p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2} \frac{\partial f}{\partial \theta} + \left(\frac{p_\theta^2}{r^3} + \frac{p_\phi^2}{r^3 \sin^2 \theta} - \frac{\partial \phi}{\partial r} \right) \frac{\partial f}{\partial p_r} + \frac{p_\phi^2 \cos \theta}{r^2 \sin^3 \theta} \frac{\partial f}{\partial p_\theta} = 0} \quad (4.272)$$

We are now going to take the moments of this equation. As a warm-up, let compute zeroth-order moment of f and try to relate it to the mass density. For this, we need to express the volume element in momentum space as a function of the volume element in velocity space. Using the previous relations, we have

$$d^3 p = dp_r dp_\theta dp_\phi = dv_r r dv_\theta r \sin \theta dv_\phi \quad (4.273)$$

$$\int_{\mathbb{R}^3} f d^3 p = r^2 \sin \theta \int_{\mathbb{R}^3} f d^3 v = r^2 \sin \theta \rho(r) \quad (4.274)$$

Let's now compute the first-order moment

$$\int_{\mathbb{R}^3} p_r [CBE] f d^3 p = 0 \quad (4.275)$$

Looking at the form of the CBE we derived above, we see that it contains 4 different terms that we will treat one by one.

$$(1) = \int_{\mathbb{R}^3} p_r^2 \frac{\partial f}{\partial r} d^3 p = \frac{\partial}{\partial r} \left(\int_{\mathbb{R}^3} p_r^2 f d^3 p \right) = \frac{\partial}{\partial r} \left(r^2 \sin \theta \int_{\mathbb{R}^3} p_r^2 f d^3 v \right) \quad (4.276)$$

We finally obtain

$$(1) = \frac{\partial}{\partial r} \left(r^2 \sin \theta \overline{\rho p_r^2} \right) \quad (4.277)$$

which can be used to define the variance of the radial momentum $\overline{p_r^2}$. The second term writes

$$(2) = \int_{\mathbb{R}^3} \frac{p_r p_\theta}{r^2} \frac{\partial f}{\partial \theta} d^3 p = \frac{\partial}{\partial \theta} \left(\int_{\mathbb{R}^3} \frac{p_r p_\theta}{r^2} f d^3 p \right) = \frac{\partial}{\partial \theta} \left(\sin \theta \int_{\mathbb{R}^3} p_r p_\theta f d^3 v \right) \quad (4.278)$$

We finally obtain

$$(2) = \frac{\partial}{\partial \theta} \left(\sin \theta \overline{\rho p_r p_\theta} \right) \quad (4.279)$$

which can be used to define the cross-correlation term $\overline{p_r p_\theta}$. The third term can be simplified using an integration by part for p_r

$$(3) = \int_{\mathbb{R}^3} \left(\frac{p_\theta^2}{r^3} + \frac{p_\phi^2}{r^3 \sin^2 \theta} - \frac{\partial \phi}{\partial r} \right) p_r \frac{\partial f}{\partial p_r} dp^3 = - \int_{\mathbb{R}^3} \left(\frac{p_\theta^2}{r^3} + \frac{p_\phi^2}{r^3 \sin^2 \theta} - \frac{\partial \phi}{\partial r} \right) f dp^3 \quad (4.280)$$

This leads again to

$$(3) = -r^2 \sin \theta \rho(r) \left(\frac{\overline{p_\theta^2}}{r^3} + \frac{\overline{p_\phi^2}}{r^3 \sin^2 \theta} - \frac{\partial \phi}{\partial r} \right) \quad (4.281)$$

which introduces the other variance terms $\overline{p_\theta^2}$ and $\overline{p_\phi^2}$. The fourth term vanishes because we can integrate directly the partial derivative of f . Before we collect the 3 non-vanishing terms into a single equation, we want to express the momentum statistics as a function of the velocity statistics. We have simply

$$\overline{p_r^2} = \overline{v_r^2} \quad \text{and} \quad \overline{p_r p_\theta} = r \overline{v_r v_\theta} \quad \text{and} \quad \overline{p_\theta^2} = r^2 \overline{v_\theta^2} \quad \text{and} \quad \overline{p_\phi^2} = r^2 \sin^2 \theta \overline{v_\phi^2} \quad (4.282)$$

We can now write the final form of the Jeans equation for our relatively general spherical system as

$$\boxed{\frac{\partial}{\partial r} \left(r^2 \rho \overline{v_r^2} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(r \sin \theta \rho \overline{v_r v_\theta} \right) - r^2 \rho(r) \left(\frac{\overline{v_\theta^2}}{r} + \frac{\overline{v_\phi^2}}{r} - \frac{\partial \phi}{\partial r} \right) = 0} \quad (4.283)$$

We will now simplify the problem further assuming first that the symmetries of f allow to assume that the radial and the tangential velocities are uncorrelated so that $\overline{v_r v_\theta} = 0$. Furthermore, we are looking for a static equilibrium solution for which all the mean velocities are zero. We can then replace everywhere $\overline{v_r^2} = \sigma_r^2$, $\overline{v_\theta^2} = \sigma_\theta^2$ and $\overline{v_\phi^2} = \sigma_\phi^2$. We can develop also

$$\frac{\partial}{\partial r} \left(r^2 \rho \sigma_r^2 \right) = 2r \rho \sigma_r^2 + r^2 \frac{\partial}{\partial r} \left(\rho \sigma_r^2 \right) \quad (4.284)$$

The Jeans equation becomes

$$\frac{\partial}{\partial r} \left(\rho \sigma_r^2 \right) + \frac{\rho(r)}{r} \left(2\sigma_r^2 - \sigma_\theta^2 - \sigma_\phi^2 \right) = -\rho(r) \frac{\partial \phi}{\partial r} \quad (4.285)$$

We define the anisotropy parameter

$$\boxed{\beta(r) = 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2}} \quad (4.286)$$

which helps simplifying even more Jeans equation into

$$\boxed{\frac{\partial}{\partial r} \left(\rho \sigma_r^2 \right) + \frac{2\beta(r)}{r} \rho \sigma_r^2 = -\rho \frac{\partial \phi}{\partial r}} \quad (4.287)$$

We can describe a few interesting asymptotic regimes.

- Isotropic orbits: the velocity dispersions are all equal in all directions, resulting into $\beta = 0$. In this case, the Jeans equation is identical to the hydrostatic equilibrium equation we derived and solved in Chapter 2.
- Purely radial orbits: this corresponds to $\sigma_\theta = \sigma_\phi = 0$ and therefore $\beta = 1$. As we show below, the Jeans equation is easily solvable in this case. It corresponds to the external regions of galactic halos where accretion flows are mostly radial.

- Nearly circular orbits: not to be confused with disks, that we will describe in the next section. Here, we have $\overline{v_\theta} = 0$ but $\sigma_\theta \gg \sigma_r$. In this case, β can become negative and very large.

The general strategy here is to assume a prescribed form for $\beta(r)$ and solve for Jeans equation to find the corresponding radial velocity dispersion profile. An interesting case is obtained with $\beta = \text{constant}$, for which the solution is (left to the reader as an exercise)

$$\rho\sigma_r^2 = r^{-2\beta} \int_r^{+\infty} x^{2\beta} \rho(x) \frac{\partial\phi}{\partial x} dx \quad (4.288)$$

Once we have found $\sigma_r(r)$, we can then deduce f using the Maxwell-Boltzmann model with $f \propto \exp(-v_r^2/2\sigma_r^2)$.

4.6.2 Jeans equation for disks and the asymmetric drift

We now derive Jeans equation for axisymmetric systems for which we obviously use cylindrical coordinates (r, θ, z) . We restrict ourselves to distribution functions of the form $f(r, z, v_r, v_\theta, v_z)$. The Lagrangian is here again $\mathcal{L} = K - V = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}\dot{z}^2 - \phi(r, z)$ and the momentum variables are

$$p_r = \dot{r} = v_r \quad \text{and} \quad p_\theta = r^2\dot{\theta} = rv_\theta \quad \text{and} \quad p_z = \dot{z} = v_z \quad (4.289)$$

The relation between the momentum space volume element and the velocity space volume element is now given by

$$d^3p = dp_r dp_\theta dp_z = r dv_r dv_\theta dv_z \quad (4.290)$$

We can now directly write the stationary CBE in cylindrical coordinates

$$\boxed{p_r \frac{\partial f}{\partial r} + p_z \frac{\partial f}{\partial z} + \left(\frac{p_\theta^2}{r^3} - \frac{\partial\phi}{\partial r} \right) \frac{\partial f}{\partial p_r} - \frac{\partial\phi}{\partial z} \frac{\partial f}{\partial p_z} = 0} \quad (4.291)$$

We first take the moment of the CBE multiplied by p_r . We get the radial momentum conservation equation

$$\frac{\partial}{\partial r} (r\rho\overline{v_r^2}) + \frac{\partial}{\partial z} (r\rho\overline{v_r v_z}) + r\rho \left(\frac{\partial\phi}{\partial r} - \frac{\overline{v_\theta^2}}{r} \right) = 0 \quad (4.292)$$

It can be simplified into

$$\frac{\partial}{\partial r} (\rho\overline{v_r^2}) + \frac{\partial}{\partial z} (\rho\overline{v_r v_z}) + \rho \left(\frac{\partial\phi}{\partial r} + \frac{\overline{v_r^2} - \overline{v_\theta^2}}{r} \right) = 0 \quad (4.293)$$

We then take the moment of the CBE multiplied by p_θ . We get the tangential momentum conservation equation

$$\frac{\partial}{\partial r} (r^2\rho\overline{v_r v_\theta}) + \frac{\partial}{\partial z} (r^2\rho\overline{v_z v_\theta}) = 0 \quad (4.294)$$

We finally take the moment of the CBE multiplied by p_z . We get the vertical momentum conservation equation

$$\frac{\partial}{\partial r} (r\rho\overline{v_r v_z}) + \frac{\partial}{\partial z} (r\rho\overline{v_z^2}) + r\rho \frac{\partial\phi}{\partial z} = 0 \quad (4.295)$$

We now assume that the underlying distribution function exhibits no cross-correlation between the different velocity components

$$\overline{v_r v_\theta} = \overline{v_z v_\theta} = \overline{v_r v_z} = 0 \quad (4.296)$$

This is usually the case if the distribution function f is even in v_r and v_z , like for example the Schwarzschild distribution function. We are left with only two equations which now read

$$\frac{\partial}{\partial r} (\rho \overline{v_r^2}) + \rho \left(\frac{\partial \phi}{\partial r} + \frac{\overline{v_r^2} - \overline{v_\theta^2}}{r} \right) = 0 \quad (4.297)$$

and

$$\frac{\partial}{\partial z} (\rho \overline{v_z^2}) + \rho \frac{\partial \phi}{\partial z} = 0 \quad (4.298)$$

If we have no vertical bulk velocity $\overline{v_z} = 0$, we finally obtain the vertical hydrostatic equilibrium equation as

$$\boxed{\frac{\partial}{\partial z} (\rho \sigma_z^2) + \rho \frac{\partial \phi}{\partial z} = 0} \quad (4.299)$$

For a stable disk, it is reasonable to assume no bulk radial velocity so that $\overline{v_r} = 0$, but we have obviously a non-zero mean tangential velocity. The radial equilibrium equation thus writes

$$\boxed{\frac{1}{\rho} \frac{\partial}{\partial r} (\rho \sigma_r^2) + \frac{\partial \phi}{\partial r} + \frac{\sigma_r^2 - \sigma_\theta^2}{r} = \frac{\overline{v_\theta^2}}{r}} \quad (4.300)$$

If we write this equation in the midplane, we recognise $v_{\text{circ}}^2 = r \frac{\partial \phi}{\partial r}$. We can re-arrange the previous equation as

$$\overline{v_\theta^2} - v_{\text{circ}}^2 = \frac{r}{\rho} \frac{\partial}{\partial r} (\rho \sigma_r^2) + \sigma_r^2 - \sigma_\theta^2 \quad (4.301)$$

Re-arranging terms one last time, we finally obtain

$$v_{\text{circ}}^2 - \overline{v_\theta^2} = \sigma_r^2 \left(\frac{\sigma_\theta^2}{\sigma_r^2} - 1 - \frac{\partial \ln (\rho \sigma_r^2)}{\partial \ln r} \right) \quad (4.302)$$

This equation is known as *the asymmetric drift* equation. Its interpretation is crucial to describe the kinematics of stars in spiral galaxies. We can exploit the properties of the epicycle ellipse around the guiding center in thin disks to compute

$$\frac{\sigma_\theta^2}{\sigma_r^2} = \frac{4\Omega_{\text{circ}}^2}{\kappa^2} \simeq 2 \quad (4.303)$$

for a flat rotation curve. We also assume that the radial pressure is always a weakly decreasing function of radius. We get

$$\boxed{v_{\text{circ}}^2 - \overline{v_\theta^2} \simeq \sigma_r^2} \quad (4.304)$$

This very powerful result states that stars rotate at a mean tangential velocity always smaller than the circular velocity, and that this difference is (almost) equal to the radial velocity dispersion. In the Milky Way, old stars have a rather large velocity dispersion, around 100 km/s, so that their tangential velocity is significantly different than the one of the young stars, who have a velocity dispersion as low as 10 km/s. The circular velocity of the Milky Way at the solar radius being around 200 km/s, we conclude that young stars rotate at $\overline{v_\theta} \simeq 200$ km/s, while old stars rotate at only $\overline{v_\theta} \simeq 170$ km/s. This validates the Schwarzschild model, but only for young stars.

4.7 Jeans instability and Landau damping

We have now extensively discussed various equilibrium solutions both in spherical and cylindrical geometries. As always in this course, we would like now to test whether these equilibrium solutions are stable. In this section, we describe the Jeans instability for collisionless systems, for which the equilibrium solution is a homogeneous and uniform background, while the next section is devoted to the stability analysis of disks. As always, we start with an equilibrium distribution function $f_0(\mathbf{x}, \mathbf{v})$ that satisfies the stationary CBE.

$$\mathbf{v} \cdot \nabla f_0 - \nabla \phi_0 \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0 \quad (4.305)$$

where the potential satisfies Poisson equation and the equilibrium density is related to the equilibrium distribution function by

$$\rho_0(\mathbf{x}) = \int_{\mathbb{R}^3} f_0(\mathbf{x}, \mathbf{v}) d^3v \quad (4.306)$$

Similarly to the derivation of Jeans instability for collisional fluids, we consider here also a uniform equilibrium density field and a uniform distribution function $f_0(\mathbf{v})$ that depends only on \mathbf{v} but not on \mathbf{x} . We end up having a similar problem, namely our equilibrium gravitational potential is ill-defined, as a uniform density field over an infinite domain results in a diverging potential. We need to adopt the same strategy than for the collisional fluid case, namely use Jeans swindle and re-define Poisson equation so that

$$\Delta \phi = 4\pi G (\rho - \rho_0) \quad (4.307)$$

This gives us both $\phi_0 = 0$ and $\nabla \phi_0 = 0$ which satisfies trivially the stationary CBE. We now perturb our uniform equilibrium solution as follows

$$f = f_0 + \delta f \quad \text{and} \quad \rho = \rho_0 + \delta \rho \quad \text{and} \quad \phi = 0 + \delta \phi \quad (4.308)$$

Injecting these perturbations into the CBE (not stationary this time of course) we get

$$\frac{\partial}{\partial t} (f_0 + \delta f) + \mathbf{v} \cdot \nabla (f_0 + \delta f) - \nabla (\delta \phi) \cdot \frac{\partial}{\partial \mathbf{v}} (f_0 + \delta f) = 0 \quad (4.309)$$

Since f_0 is constant and uniform, we can remove the corresponding time and space derivatives. We now also linearise the CBE by neglecting the quadratic term $\nabla (\delta \phi) \cdot (\delta f)$. We obtain the linearised CBE

$$\boxed{\frac{\partial}{\partial t} (\delta f) + \mathbf{v} \cdot \nabla (\delta f) - \nabla (\delta \phi) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0} \quad (4.310)$$

This equation has to be supplemented with the two already linearised equations for $\delta \rho$ and $\delta \phi$

$$\Delta (\delta \phi) = 4\pi G (\delta \rho) \quad \text{and} \quad \delta \rho = \int_{\mathbb{R}^3} \delta f d^3v \quad (4.311)$$

We are now considering solutions of the form of planar waves, whose wave vector \mathbf{k} is considered, without loss of generality, aligned with the x-axis.

$$\delta f = \hat{f} \exp^{i(kx - \omega t)} \quad \text{and} \quad \delta \phi = \hat{\phi} \exp^{i(kx - \omega t)} \quad \text{and} \quad \delta \rho = \hat{\rho} \exp^{i(kx - \omega t)} \quad (4.312)$$

Note that the amplitudes of the planar waves are analogous to the Fourier coefficients of a general function $\delta f(x, t)$. Injecting our planar waves Ansatz into the CBE leads to the dispersion relation

$$-i\omega \hat{f} + i v_x k \hat{f} - ik \hat{\phi} \frac{\partial f_0}{\partial v_x} = 0 \quad (4.313)$$

Re-arranging terms we finally get

$$\hat{f} = \frac{k \frac{\partial f_0}{\partial v_x}}{k v_x - \omega} \hat{\phi} \quad (4.314)$$

Poisson equation can also be solved using our Ansatz as

$$-k^2 \hat{\phi} = 4\pi G \hat{\rho} \quad \text{and} \quad \hat{\rho} = \int_{\mathbb{R}^3} \hat{f} d^3 v \quad (4.315)$$

If we inject the equation for \hat{f} into the last one, we get

$$\hat{\rho} = \hat{\phi} \int_{\mathbb{R}^3} \frac{k \frac{\partial f_0}{\partial v_x}}{k v_x - \omega} d^3 v \quad (4.316)$$

In the last equation, we could remove $\hat{\phi}$ from the integral, because it depends only on (Fourier) space coordinates. Using Poisson equation and requesting the amplitude of our waves to be non-zero, we finally get the dispersion relation

$$1 = -\frac{4\pi G}{k^2} \int_{\mathbb{R}^3} \frac{k \frac{\partial f_0}{\partial v_x}}{k v_x - \omega} d^3 v \quad (4.317)$$

We can simplify this equation introducing a function we are already familiar with, namely the one-dimensional distribution function defined by

$$F_0(v_x) = \int_{\mathbb{R}^2} f_0(v_x, v_y, v_z) dv_y dv_z \quad (4.318)$$

The dispersion relation becomes

$$1 = -\frac{4\pi G}{k^2} \int_{\mathbb{R}} \frac{k \frac{\partial F_0}{\partial v_x}}{k v_x - \omega} dv_x \quad (4.319)$$

We immediately see that the denominator can vanish if the particle velocity is in resonance with the wave, namely when $v_x \simeq \omega/k$. In this case, especially if ω is a real number, the integral is ill-defined. We now assume that f_0 is a Maxwell-Boltzman distribution so that

$$f_0 = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} \exp^{-\frac{v^2}{2\sigma^2}} \quad \text{and} \quad F_0 = \frac{\rho_0}{(2\pi\sigma^2)^{1/2}} \exp^{-\frac{v_x^2}{2\sigma^2}} \quad (4.320)$$

where σ is the velocity dispersion characterising the equilibrium solution f_0 . We immediately get

$$\frac{\partial F_0}{\partial v_x} = \frac{\rho_0}{(2\pi\sigma^2)^{1/2}} \left(-\frac{v_x}{\sigma^2} \right) \exp^{-\frac{v_x^2}{2\sigma^2}} \quad (4.321)$$

We introduce the normalised velocity $u = v_x/\sigma$ and get for the dispersion relation

$$1 = \frac{4\pi G \rho_0}{k^2 \sigma^2} \int_{\mathbb{R}} \frac{k \sigma u}{k \sigma u - \omega} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \quad (4.322)$$

which can be further simplified into

$$1 = \frac{4\pi G \rho_0}{k^2 \sigma^2} \left(1 + \omega \int_{\mathbb{R}} \frac{1}{k \sigma u - \omega} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \right) \quad (4.323)$$

Note that in the general case, $\omega = \omega_r + i\omega_i$ is a complex number. We define another complex number

$$Z(\omega) = Z_r + iZ_i = \int_{\mathbb{R}} \frac{1}{k\sigma u - \omega} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \quad (4.324)$$

The dispersion relation takes now the very compact form

$$1 = \frac{k_J^2}{k^2} (1 + \omega Z(\omega)) \quad (4.325)$$

where we have introduced the *Jeans wave number*

$$k_J^2 = \frac{4\pi G\rho_0}{\sigma^2} \quad (4.326)$$

Interestingly, it is identical to the Jeans wave number that we found in the collisional fluid case, with the sound speed replaced by the velocity dispersion. Since ω is a complex number, we need to consider the three possible cases:

- $\omega_i > 0$: growing amplitude, unstable waves
- $\omega_i = 0$: constant amplitude, travelling waves
- $\omega_i < 0$: damped waves

4.7.1 Collisionless Jeans instability

We consider here the unstable case with $\omega_i > 0$. The integral over u does not have a pole, namely a possible location where the integral might diverge. Indeed, if $\omega_i > 0$, the pole where the resonance occurs is off the x -axis. The complex number $Z(\omega)$ can be expressed as

$$Z(\omega) = \int_{\mathbb{R}} \frac{k\sigma u - \omega_r + i\omega_i}{(k\sigma u - \omega_r)^2 + \omega_i^2} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \quad (4.327)$$

We thus deduce

$$Z_r = \int_{\mathbb{R}} \frac{k\sigma u - \omega_r}{(k\sigma u - \omega_r)^2 + \omega_i^2} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \quad (4.328)$$

and

$$Z_i = \int_{\mathbb{R}} \frac{\omega_i}{(k\sigma u - \omega_r)^2 + \omega_i^2} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \quad (4.329)$$

We see that both integrals are well defined and converge for all possible values of ω_r . We can also express the dispersion relation more explicitly as

$$1 = \frac{k_J^2}{k^2} [1 + \omega_r Z_r - \omega_i Z_i + i(\omega_i Z_r + \omega_r Z_i)] \quad (4.330)$$

We can compute directly the imaginary component of the dispersion relation

$$\omega_i Z_r + \omega_r Z_i = \omega_i \int_{\mathbb{R}} \frac{k\sigma u}{(k\sigma u - \omega_r)^2 + \omega_i^2} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \quad (4.331)$$

We find that it is zero only if $\omega_r = 0$. This means we have no travelling wave solution. Because $\omega_r = 0$, we see that Z_r is also zero because the integrand is an odd function of u . Since both ω_r and Z_r vanish, we can re-write the dispersion relation as

$$1 = \frac{k_J^2}{k^2} [1 - \omega_i Z_i] \quad (4.332)$$

which has now no imaginary component as it should. We also have in this case

$$Z_i = \int_{\mathbb{R}} \frac{\omega_i}{(k\sigma u)^2 + \omega_i^2} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du > 0 \quad (4.333)$$

Interestingly, since $\omega_i Z_i > 0$, we conclude that $k < k_J$. The unstable regime corresponds to wavelengths larger than the Jeans length, exactly like for collisional fluids. In the very large wavelength limit, we can compute exactly the corresponding instability growth rate. Indeed, we write the dispersion relation as

$$1 = \frac{k_J^2}{k^2} \left[1 - \int_{\mathbb{R}} \frac{1}{1 + (k\sigma u/\omega_i)^2} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \right] \quad (4.334)$$

For very small wave numbers with $k \ll k_J$, we can perform a Taylor expansion of the integrand as follows

$$\frac{1}{1 + (k\sigma u/\omega_i)^2} \simeq 1 - (k\sigma u/\omega_i)^2 \quad (4.335)$$

Because the Gaussian satisfies these two relations

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du = 1 \quad \text{and} \quad \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} u^2 \exp^{-\frac{u^2}{2}} du = 1, \quad (4.336)$$

we end up with the following dispersion relation

$$1 \simeq \frac{k_J^2}{k^2} \left[1 - \left(1 - \frac{k^2 \sigma^2}{\omega_i^2} \right) \right] = \frac{k_J^2 \sigma^2}{\omega_i^2} \quad (4.337)$$

We finally obtain for the growth rate in the limit $k \ll k_J$

$$\boxed{\omega_i \simeq \omega_{\text{ff}} = k_J \sigma = \sqrt{4\pi G \rho_0}} \quad (4.338)$$

which is the inverse of the free-fall time, again identical to the collisional fluid result.

4.7.2 Absence of propagating wave solutions

We are now looking for propagating wave solutions, for which $\omega_i = 0$. If $\omega_r \neq 0$, then we have a pole on the x-axis where the previous integrands diverge. We have in particular

$$Z(\omega) = \int_{\mathbb{R}} \frac{1}{k\sigma u - \omega_r} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \quad (4.339)$$

We would like to use Cauchy formula, which, for an analytic complex function $f(z)$ and for a positively oriented closed contour that contains the pole z_0 , writes

$$\int_C \frac{f(z)}{z - z_0} dz = i\pi f(z_0) \quad (4.340)$$

The natural choice would be to use a contour defined by the x-axis and the upper half-circle with infinite radius. The problem is that the Gaussian function is ill-defined for large imaginary numbers. It is better to use *the Landau contour* as the union of the entire x-axis and a half-circle with a vanishingly small radius around the pole. In this case, we get

$$Z(\omega) = \mathcal{PV} \left\{ \int_{\mathbb{R}} \frac{1}{k\sigma u - \omega_r} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \right\} + i\pi \frac{1}{\sqrt{2\pi}} \frac{\exp^{-\frac{\omega_r^2}{2\sigma^2 k^2}}}{k\sigma} \quad (4.341)$$

where \mathcal{PV} stands for the *Principal Value* of the integral. We see that we obtain

$$Z_r = \mathcal{PV} \left\{ \int_{\mathbb{R}} \frac{1}{k\sigma u - \omega_r} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{u^2}{2}} du \right\} \quad (4.342)$$

and

$$Z_i = \pi \frac{1}{\sqrt{2\pi}} \frac{\exp^{-\frac{\omega_r^2}{2\sigma^2 k^2}}}{k\sigma} \quad (4.343)$$

In this regime where $\omega_i = 0$, the dispersion relation writes

$$1 = \frac{k_J^2}{k^2} [1 + \omega_r Z_r + i(\omega_r Z_i)] \quad (4.344)$$

We see here again a problem: if $\omega_r \neq 0$ then the dispersion relation contains a non-vanishing imaginary component, because $Z_i > 0$. We must necessarily have $\omega_r = 0$, which means *there are no travelling wave solutions*. This result is in striking contrast with the collisional fluid case, where we found travelling sound waves. For collisionless fluids, there are no sound waves, only unstable waves with $\omega_i > 0$ or damped waves with $\omega_i < 0$. Interestingly, the trivial case $\omega = 0$ that we just found corresponds exactly to $k = k_J$, the Jeans wavenumber.

4.7.3 Landau damping

We can solve for the dispersion relation for damped waves with $\omega_i < 0$ and $k > k_J$. The presence of the singularity at $\omega = vk$ shows that particles moving at a speed close to the wave speed (also called resonant particles) play an important role in the damping of self-gravitating waves. It is however quite difficult to do in practice and requires a lot of virtuosity in the complex plane. We will use instead in this paragraph a different path that will clearly expose the physical process known as *Landau damping*. We consider the gravitation acceleration of the planar wave perturbation we introduced above, with

$$\delta g(x, t) = g_0 \cos(kx - \omega t) \quad (4.345)$$

For simplicity, we have set the phase and the amplitude so that $g = g_0 > 0$ for $x = t = 0$. We then consider one particle with initial position $x = 0$ at $t = 0$ and initial velocity v_0 . The initial particle velocity is drawn from the equilibrium Maxwell-Boltzmann distribution with variance σ . Note that δg is a small perturbation of the equilibrium gravitational acceleration. We don't expect the trajectory of the particle to be strongly affected by this perturbation. We thus have to first order that the trajectory is a straight line with

$$v(t) \simeq v_0 \quad \text{and} \quad x(t) \simeq v_0 t \quad (4.346)$$

The gravitational acceleration that this particle will feel is therefore given to first order by

$$\frac{dv}{dt} \simeq g_0 \cos(kv_0 - \omega)t \quad (4.347)$$

We can now compute a better, second-order solution for the particle velocity as

$$v(t) = v_0 + \int_0^t g_0 \cos(kv_0 - \omega)t dt = v_0 + g_0 \frac{\sin(kv_0 - \omega)t}{kv_0 - \omega} \quad (4.348)$$

Note that if the particle moves initially at exactly the wave speed with $kv_0 = \omega$, then the trajectory is a parabola with

$$v(t) = v_0 + g_0 t \quad \text{and} \quad x(t) = v_0 t + \frac{1}{2} g_0 t^2 \quad (4.349)$$

This sets the time scale over which the particle distribution will be accelerated so much that the perturbation cannot be considered as small anymore.

$$t_0 = \frac{\sigma}{g_0} \quad (4.350)$$

We will use this time scale in what follows as the *maximum time scale* over which the wave can propagate. If $kv_0 \neq \omega$, we can integrate the velocity and obtain the trajectory as

$$x(t) = v_0 t + \int_0^t g_0 \frac{\sin(kv_0 - \omega)t}{kv_0 - \omega} dt = v_0 t + g_0 \frac{1 - \cos(kv_0 - \omega)t}{(kv_0 - \omega)^2} \quad (4.351)$$

We see that the particle trajectory is modulated by the underlying wave at a much lower frequency $\Omega = kv_0 - \omega$. The trajectory of resonant particles will feature a particularly low frequency, ultimately going to zero as $kv_0 \rightarrow \omega$. We now compute the instantaneous power gained by the particle due to the gravitational force.

$$P(t) = m\delta g(t)v(t) \quad (4.352)$$

For the gravity acceleration, we have to use also a second-order solution as

$$\delta g(t) = g_0 \cos(kx(t) - \omega t) = g_0 \cos(kv_0 t + k\delta x - \omega t) \quad (4.353)$$

We perform a Taylor expansion and get the second-order solution for the acceleration on the particle as

$$\delta g(t) \simeq g_0 \cos(kv_0 - \omega)t - g_0 k\delta x \sin(kv_0 - \omega)t \quad (4.354)$$

Recall that we already found

$$\delta x = g_0 \frac{1 - \cos(kv_0 - \omega)t}{(kv_0 - \omega)^2} \quad \text{and} \quad v(t) = v_0 + g_0 \frac{\sin(kv_0 - \omega)t}{kv_0 - \omega} \quad (4.355)$$

Injecting this in the instantaneous power will produce several terms, that we will not write here. The main result is that the instantaneous power is an oscillating function of time. Particles gain and loose energy, in sync with the frequency of the wave Ω , similarly to a surfer riding a wave. Following what we did in the Chapter on radiative transfer, it makes more sense to average the power over one period to get an idea of the secular energy gain or loss of the particle. The period is here given be

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{kv_0 - \omega} \quad (4.356)$$

Only one term will survive the averaging procedure, namely

$$P_0 = \frac{1}{T} \int_0^T P(t) dt = -\frac{m}{2} \frac{kg_0^3}{(kv_0 - \omega)^3} \quad (4.357)$$

For reasons that will become clear later, we re-write this result as

$$P_0 = \frac{m}{4} g_0^3 \frac{\partial \phi}{\partial v_0} \quad \text{with} \quad \phi(v_0) = \frac{1}{(kv_0 - \omega)^2} \quad (4.358)$$

This result requires some discussion. We see that particles with $kv_0 < \omega$ will gain energy ($P_0 > 0$) while particle with $kv_0 > \omega$ will loose energy ($P_0 < 0$) as they ride the gravitational planar wave. The closer the particles are to the resonant velocity $kv_0 = \omega$, the more energy they gain (or loose). In fact, the power diverges for resonant particles. This problem can be regularised by noticing that the oscillation frequency Ω for these waves is so small that one

cannot average over at least one period when the maximum allowed time is reached. In other words, the particles with maximum energy gain and loss will be those within

$$k\Delta v = |kv_0 - \omega| = \frac{2\pi}{t_0} = \frac{2\pi g_0}{\sigma} \quad (4.359)$$

from the wave frequency ω . We can regularise the previous equation for the power using the Lorentz line profile we derived in Chapter 3. Indeed, we can replace the previous diverging formula with a regularised version as follows

$$P_0 = \frac{m}{4} g_0^3 \frac{\partial \phi}{\partial v_0} \quad \text{with} \quad \phi(v_0) = \frac{1}{(kv_0 - \omega)^2 + k^2 \Delta v^2} \quad (4.360)$$

We can now integrate the total energy lost or gained by the entire population of particle using the Maxwell-Boltzmann distribution. We use a smart integration by part to obtain

$$P_{\text{tot}} = \int_{\mathbb{R}} P_0 F_0(v_0) dv_0 = \frac{1}{4} g_0^3 \int_{\mathbb{R}} \frac{\partial \phi}{\partial v_0} F_0(v_0) dv_0 = -\frac{1}{4} g_0^3 \int_{\mathbb{R}} \frac{\partial F_0}{\partial v_0} \phi(v_0) dv_0 \quad (4.361)$$

Note that the particle mass disappeared because it has been absorbed by the distribution function F_0 as it is normalised to the mass density ρ_0 . Remember also that F_0 is a one-dimensional Maxwell-Boltzmann distribution with

$$\frac{\partial F_0}{\partial v_0} = -\frac{\rho_0}{\sqrt{2\pi}} \frac{v_0}{\sigma^3} \exp^{-\frac{v_0^2}{2\sigma^2}} \quad (4.362)$$

Because ϕ is a very narrow line, like a typical Lorentz profile, we can evaluate the integrand at the resonance $kv_0 = \omega$ and obtain

$$P_{\text{tot}} \simeq -\frac{1}{4} g_0^3 \frac{\partial F_0(\omega/k)}{\partial v_0} \int_{-\infty}^{+\infty} \phi(v_0) dv_0 = -\frac{1}{4} g_0^3 \frac{\partial F_0(\omega/k)}{\partial v_0} \frac{\pi}{k^2 \Delta v} \quad (4.363)$$

Injecting the various terms we have computed above, we finally get

$$P_{\text{tot}} = \frac{1}{8} \frac{\rho_0 g_0^2}{\sigma^2} \frac{\omega}{k^2} \frac{1}{\sqrt{2\pi}} \exp^{-\frac{\omega^2}{2k^2\sigma^2}} \quad (4.364)$$

Because we have integrated over all particles in the Maxwell-Boltzmann distribution, we obtain here the energy gained by particles per unit volume. Note that it is positive. This is because we have more particles with $kv_0 < \omega$ than particles with $kv_0 > \omega$ in the Maxwell-Boltzmann distribution. Because the kinetic energy of the overall particle distribution is increasing, the potential energy in the gravitational planar wave has to decrease. Hence the name *Landau damping*. We know from a previous section that the potential energy density of the gravitational planar wave (integrated here in time over one period and in space over one wavelength) is given by

$$e_{\text{pot}} = -\frac{1}{8\pi G} \frac{1}{4} g_0^2. \quad (4.365)$$

We can immediately deduce the damping rate as

$$\omega_{\text{damp}} = -\frac{P_{\text{tot}}}{e_{\text{pot}}} = \omega \frac{k_J^2}{k^2} \frac{\exp^{-\frac{\omega^2}{2k^2\sigma^2}}}{\sqrt{2\pi}} \quad (4.366)$$

Remember that we are in the regime $k > k_J$, so that this damping rate is usually quite slow compared to the wave frequency ω . For very small wavelength, the damping time scale becomes

so long that it can be ignored and the gravitational planar waves behave like sound waves. On the other hand, damping becomes stronger as one approaches the Jeans length. For collisionless fluids, particles can remain close to the resonance long enough so that they can gain energy from the wave. In the strong collisional case, however, collisions are so frequent that resonant particles are constantly kicked out of their surf board. They cannot stay at the top of the wave and pump enough energy out of it. This is why there is no damping for self-gravitating sound waves in the collisional fluid case.

4.8 Toomre instability in self-gravitating disks

In this section, we describe a very important physical process in astrophysics, namely the instability of self-gravitating disks under fragmentation. This process is at the origin of the formation of molecular clouds and spiral arms in galaxies, as well as the formation of giant planets and spiral structures in massive proto-planetary disks. Note also that the instability analysis we present here applies for both gas and stellar disks. We will first draw a simple picture for this disk instability based on the competition of several well identified processes. We will then derive more rigorously the dispersion relation for self-gravitating waves in disks.

4.8.1 Toomre parameter

In this section, we consider again razor-thin disks for which the volume density is given by

$$\rho(r, z) = \Sigma(r)\delta(z) \quad (4.367)$$

We are now consider a small perturbation at the surface of the disk of size λ . This perturbation can be seen as a small circle of radius λ encompassing a small amount of gas and stars. We can compute the mass in this clump as

$$M_{\text{clump}} = \Sigma(r)\pi\lambda^2 \quad (4.368)$$

The question we ask now is the following: *can this clump collapse under its own gravity?* For this, we consider two main effects that can prevent the collapse.

- **Thermal support:** thermal or turbulent random motions of gas and stars can provide support against the collapse of the cloud. We have seen in Chapter 2 that the virial theorem can be used to check if a system should collapse or expand. The condition for the collapse is given by the virial parameter $\alpha_{\text{vir}} < 1$

$$\frac{GM_{\text{clump}}}{\lambda} > c_s^2 \quad (4.369)$$

Note here that c_s is the gas sound speed, but it could be replaced by the star velocity dispersion or the turbulence velocity dispersion. Injecting the equation for the mass of the clump, we get the condition

$$\lambda > \lambda_J = \frac{c_s^2}{\pi G \Sigma} \quad (4.370)$$

where the critical cloud size we found is called the *two-dimensional Jeans length*. It is different from the Jeans length we found for three-dimensional homogeneous systems.

- **Centrifugal support:** our small clump is also rotating in the frame of the guiding centre, following the epicycle ellipse. Particles of gas or stars are experiencing a centrifugal force

in this frame that tends to pull the cloud apart, working against gravity. We can write the condition for the collapse as

$$\frac{GM_{\text{clump}}}{\lambda^2} > \frac{\dot{x}^2}{\lambda} \quad (4.371)$$

In other words, the gravitational acceleration of the cloud exceeds the centrifugal force due to the epicycle rotation velocity. We now from the previous sections that we can write $\dot{x}^2 = \kappa^2 x^2$ where we use for the epicycle ellipse $x \simeq \lambda$. This leads to

$$\frac{GM_{\text{clump}}}{\lambda^2} > \kappa^2 \lambda \quad (4.372)$$

or equivalently

$$\lambda < \lambda_T = \frac{\pi G \Sigma}{\kappa^2} \quad (4.373)$$

where the second critical length we found is called *the Toomre length*.

Combining these two results, we see that the conditions for a cloud of size λ to collapse is given by $\lambda_J < \lambda < \lambda_T$. In other words, the disk will be unstable against fragmentation, if a gap can be opened between the Jeans length and the Toomre length or $\lambda_J < \lambda_T$. In order to get a stable disk, on the other hand, we just need to have $\lambda_J \geq \lambda_T$ and the gap will be closed. This condition can be turned into a stability criteria for the disk

$$\lambda_J = \frac{c_s^2}{\pi G \Sigma} \geq \lambda_T = \frac{\pi G \Sigma}{\kappa^2} \quad (4.374)$$

We introduce the Toomre parameter Q . The condition for collapse is $Q < 1$ and the condition for stability is $Q \geq 1$.

$$Q = \frac{c_s \kappa}{\pi G \Sigma} \quad (4.375)$$

4.8.2 Stability analysis of a razor-thin disk

We now switch to a more rigorous treatment of the disk instability. We will use the Euler equation for an isothermal gas. Note that one could have used instead the Jeans equations for a constant velocity dispersion stellar disk and obtained a similar result. We use here obviously a cylindrical coordinate system. The Euler equations can be written in cylindrical coordinate as follows

$$\frac{Dv_r}{Dt} = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{1}{r} \frac{\partial v_r}{\partial \theta} = \frac{v_\theta^2}{r} - \frac{\partial \phi}{\partial r} - \frac{1}{\Sigma} \frac{\partial P}{\partial r} \quad (4.376)$$

$$\frac{Dv_\theta}{Dt} = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = -\frac{v_\theta v_r}{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} - \frac{1}{\Sigma} \frac{1}{r} \frac{\partial P}{\partial \theta} \quad (4.377)$$

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r v_r \Sigma) + \frac{1}{r} \frac{\partial}{\partial \theta} (v_\theta \Sigma) = 0 \quad (4.378)$$

Note that these equations are identical to the one we derived for accretion disks. The only difference is the self-gravity of the disk. Since our gas is isothermal, we can write $P = \Sigma c_s^2$ where c_s is the isothermal sound speed. We can also write the pressure gradient term as

$$\frac{1}{\Sigma} \frac{\partial P}{\partial r} = \frac{\partial}{\partial r} (c_s^2 \log \Sigma) = \frac{\partial \Pi}{\partial r} \quad \text{where} \quad \Pi = c_s^2 \log \frac{\Sigma}{\Sigma_0} \quad (4.379)$$

The equilibrium model we consider here can be any of the equilibrium disks we have presented in this course. It is defined by zero radial velocity, a tangential velocity close to the circular

velocity and a given potential-density pair. We now perturb our equilibrium model slightly, so that the perturbed velocity field can be written as

$$v_r = 0 + \delta v_r \quad \text{and} \quad v_\theta = v_\theta^0 + \delta v_\theta \quad (4.380)$$

and the perturbed potential-density pair can be written as

$$\Sigma = \Sigma_0 + \delta \Sigma \quad \text{and} \quad \phi = \phi_0 + \delta \phi \quad (4.381)$$

Finally, the perturbed pressure can be written as

$$\Pi = c_s^2 \log \frac{\Sigma_0 + \delta \Sigma}{\Sigma_0} \simeq \Pi_0 + c_s^2 \frac{\delta \Sigma}{\Sigma_0} \quad (4.382)$$

Note that we have just linearised the pressure perturbation and obtained

$$\boxed{\delta \Pi = \frac{c_s^2}{\Sigma_0} \delta \Sigma} \quad (4.383)$$

Let's now linearise the other equations of the model. For the radial velocity we can re-write the first Euler equation, dropping all quadratic terms

$$\frac{\partial}{\partial t} (\delta v_r) + \frac{v_\theta^0}{r} \frac{\partial}{\partial \theta} (\delta v_r) = 2 \frac{v_\theta^0}{r} (\delta v_\theta) - \frac{\partial}{\partial r} (\delta \phi + \delta \Pi) \quad (4.384)$$

Note that we have also removed in the previous equation equilibrium terms using the equilibrium condition

$$\frac{(v_\theta^0)^2}{r} = \frac{\partial \phi_0}{\partial r} + \frac{\partial \Pi_0}{\partial r} \quad (4.385)$$

The tangential velocity equation can be linearised as follows

$$\frac{\partial}{\partial t} (\delta v_\theta) + \frac{\partial v_\theta^0}{\partial r} (\delta v_r) + \frac{v_\theta^0}{r} \frac{\partial}{\partial \theta} (\delta v_\theta) = -\frac{v_\theta^0}{r} (\delta v_r) - \frac{1}{r} \frac{\partial}{\partial \theta} (\delta \phi + \delta \Pi) \quad (4.386)$$

where we used the fact that the equilibrium variables are all axisymmetric. Finally, we linearise the mass conservation equation as

$$\frac{\partial}{\partial t} (\delta \Sigma) + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma_0 \delta v_r) + \frac{v_\theta^0}{r} \frac{\partial}{\partial \theta} (\delta \Sigma) + \frac{\Sigma_0}{r} \frac{\partial}{\partial \theta} (\delta v_\theta) = 0 \quad (4.387)$$

We recognise everywhere in these equations the orbital frequency of the equilibrium model $v_\theta^0 = \Omega r$. In order to facilitate the next steps, we now consider as always planar wave solutions but this time only in the tangential direction

$$\delta v_r = \widehat{v}_r(r) \exp^{i(m\theta - \omega t)}, \quad \delta v_\theta = \widehat{v}_\theta(r) \exp^{i(m\theta - \omega t)}, \quad \text{etc.} \quad (4.388)$$

This Ansatz could be interpreted as a Fourier transform in time and angular domain. We can compute time and angular derivatives easily and derive new forms for the previous equations. The radial velocity equation writes for example

$$(-i\omega + im\Omega) \widehat{v}_r - 2\Omega \widehat{v}_\theta = -\frac{\partial}{\partial r} (\widehat{\phi} + \widehat{\Pi}) \quad (4.389)$$

The tangential velocity equation becomes

$$(-i\omega + im\Omega) \widehat{v}_\theta + \left(\frac{\partial v_\theta^0}{\partial r} + \Omega \right) \widehat{v}_r = -im \frac{1}{r} (\widehat{\phi} + \widehat{\Pi}) \quad (4.390)$$

Finally, we have for the mass conservation equation

$$(-i\omega + im\Omega) \widehat{\Sigma} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma_0 \widehat{v}_r) + im \frac{\Sigma_0}{r} \widehat{v}_\theta = 0 \quad (4.391)$$

We can combine the two velocity equations, multiplying the radial one by $(-i\omega + im\Omega)$, the second one by 2Ω and adding them up, and obtain

$$\left[-(\omega - m\Omega)^2 + 2\Omega \left(\frac{\partial v_\theta^0}{\partial r} + \Omega \right) \right] \widehat{v}_r = i(\omega - m\Omega) \frac{\partial}{\partial r} (\widehat{\phi} + \widehat{\Pi}) - i2m\Omega \frac{1}{r} (\widehat{\phi} + \widehat{\Pi}) \quad (4.392)$$

In the left bracket, we recognise the epicyclic frequency

$$\kappa^2 = 2\Omega \left(r \frac{\partial \Omega}{\partial r} + 2\Omega \right) \quad (4.393)$$

The radial velocity equation can be written as

$$\left[\kappa^2 - (\omega - m\Omega)^2 \right] \widehat{v}_r = i(\omega - m\Omega) \frac{\partial}{\partial r} (\widehat{\phi} + \widehat{\Pi}) - i2m\Omega \frac{1}{r} (\widehat{\phi} + \widehat{\Pi}) \quad (4.394)$$

Following a similar strategy, we get the tangential velocity equation as

$$\left[\kappa^2 - (\omega - m\Omega)^2 \right] \widehat{v}_\theta = \frac{\kappa^2}{2\Omega} \frac{\partial}{\partial r} (\widehat{\phi} + \widehat{\Pi}) - m(\omega - m\Omega) \frac{1}{r} (\widehat{\phi} + \widehat{\Pi}) \quad (4.395)$$

It is interesting to realise that the left-hand side of the two velocity equations contains a term that can vanish. If this happens, we have a resonance and the amplitudes of the wave can diverge. This happens when the wave frequency is equal to the Lindblad resonances

$$\boxed{\omega_L = m\Omega \pm \kappa} \quad (4.396)$$

For $m = 0$ (axisymmetric perturbations), the Lindblad resonance frequency is equal to the epicyclic frequency. It means that stars who are oscillating in the radial direction at the epicyclic frequency are in sync with the radial perturbation of the potential. Similarly to Landau damping, this creates a resonance with a strong energy exchange between the stars and the wave.

4.8.3 Local disk approximation

In order to simplify the problem, we adopt now the *local disk approximation*. We consider a small portion of the disk at a specified radius r_0 , where the geometrical terms due to the cylindrical coordinates can be ignored. In mathematical terms, this corresponds to

$$\frac{\partial}{\partial r} (\widehat{\phi}) \gg \frac{\widehat{\phi}}{r}, \quad \frac{\partial}{\partial r} (\widehat{\Delta}) \gg \frac{\widehat{\Delta}}{r}, \quad \text{etc.} \quad (4.397)$$

Our equations become of course way simpler, with

$$\left[\kappa^2 - (\omega - m\Omega)^2 \right] \widehat{v}_r = i(\omega - m\Omega) \frac{\partial}{\partial r} (\widehat{\phi} + \widehat{\Pi}) \quad (4.398)$$

$$(-i\omega + im\Omega) \widehat{\Sigma} + \frac{\partial}{\partial r} (\Sigma_0 \widehat{v}_r) = 0 \quad (4.399)$$

We have omitted the tangential velocity equation for clarity. We can now Fourier transform our solution in the radial direction, since geometrical terms have been removed and our local geometry is now quasi-Cartesian with $x = r$ and $y = r_0\theta$. We therefore introduce

$$\widehat{v}_r(r) = \widetilde{v}_r \exp^{ikr}, \quad \widehat{v}_\theta(r) = \widetilde{v}_\theta \exp^{ikr}, \quad \text{etc.} \quad (4.400)$$

where the wavenumber k corresponds to k_x and the tangential mode m corresponds to $k_y r_0$. Our equations now write

$$\left[\kappa^2 - (\omega - m\Omega)^2 \right] \tilde{v}_r = -k(\omega - m\Omega) \left(\tilde{\phi} + \tilde{\Pi} \right) \quad (4.401)$$

$$(-i\omega + im\Omega) \tilde{\Sigma} = -ik\Sigma_0 \tilde{v}_r \quad (4.402)$$

Remember that the pressure perturbation is related to the density perturbation by

$$\tilde{\Pi} = \frac{c_s^2}{\Sigma_0} \tilde{\Sigma} \quad (4.403)$$

An important missing ingredient is here the gravitational potential. Remember that we found that the Fourier amplitude of the potential of a razor-thin disk is related to the Fourier amplitude of the surface density by a 2D Poisson equation

$$\tilde{\phi} = -\frac{2\pi G}{\sqrt{k_x^2 + k_y^2}} \tilde{\Sigma} \simeq -\frac{2\pi G}{|k|} \tilde{\Sigma} \quad (4.404)$$

Here, we consider only low m modes, so that it is indeed safe to assume that $k_y \ll k_x$ and $k_x \simeq k$. Injecting the values of $\tilde{\phi}$ and $\tilde{\Pi}$ as a function of $\tilde{\Sigma}$, we finally get

$$\tilde{v}_r = -\frac{(\omega - m\Omega)}{\left[\kappa^2 - (\omega - m\Omega)^2 \right]} \left(\frac{c_s^2}{\Sigma_0} - \frac{2\pi G}{|k|} \right) k \tilde{\Sigma} \quad (4.405)$$

$$\tilde{\Sigma} = \frac{\Sigma_0}{(\omega - m\Omega)} k \tilde{v}_r \quad (4.406)$$

Combining the two equations, we obtain finally the dispersion relation

$$1 = -\frac{k^2}{\left[\kappa^2 - (\omega - m\Omega)^2 \right]} \left(c_s^2 - \frac{2\pi G \Sigma_0}{|k|} \right) \quad (4.407)$$

which can be simplified as

$$\boxed{k^2 c_s^2 - 2\pi G \Sigma_0 k + \kappa^2 = (\omega - m\Omega)^2} \quad (4.408)$$

This famous result is known as *the Toomre dispersion relation*. Be careful to make the difference between the radial wave number k and the epicyclic frequency κ . It might be useful to remind the reader what is the actual form of the solutions, shown here for example for the radial velocity

$$\delta v_r = \tilde{v}_r \exp^{i(m\theta + kr - \omega t)} \quad (4.409)$$

As always, if we find that $\omega = \omega_r + i\omega_i$ has a positive imaginary part, we have an instability. This is true if the left-hand side of the dispersion relation is negative

$$k^2 c_s^2 - 2\pi G \Sigma_0 k + \kappa^2 < 0 \quad (4.410)$$

We have to find at least one real root for this degree-two polynomial which is equivalent to the (reduced) discriminant being positive

$$\Delta' = \pi^2 G^2 \Sigma_0^2 - \kappa^2 c_s^2 > 0 \quad (4.411)$$

or equivalently the Toomre parameter being less than 1.

$$Q = \frac{\kappa c_s}{\pi G \Sigma_0} < 1 \quad (4.412)$$

Interestingly, the Toomre wavelength λ_T corresponds to the smallest root k_T , while the Jeans length corresponds to the largest root k_J . The most unstable wavenumber corresponds to the minimum of the parabola which is reached for

$$k_{\min} = \frac{\pi G \Sigma_0}{c_s^2} \quad (4.413)$$

and the corresponding wave frequency reads

$$\omega = m\Omega + i\kappa \sqrt{\frac{1}{Q^2} - 1} \quad (4.414)$$

Interestingly, we see that the instability growth rate is close to the epicyclic frequency of the disk at the current location r_0 . This instability plays a major role in galactic dynamics and proto-stellar disk evolution. It explains the origin of spiral waves in quiescent disks like the Milky Way or violent fragmentation in high redshift, gas rich galaxies.

4.9 Dynamical friction

In this last section, we would like to describe collisions more explicitly and how they might affect collisionless orbits. When we have computed the relaxation time, we have estimated how much kinetic energy of the subject star was lost by two-body interactions with the other stars in the system, the so-called field stars. In this section, we would like to repeat this derivation more rigorously, but this time looking at the momentum exchange between the field stars and the subject star. Moreover, the subject star is assumed to have a mass $M \gg m$, much larger than the mass of the field stars. We can describe the gravitational collision as usual, in the frame of the centre of mass, which in this case is the frame comoving with the massive subject star. We consider field stars bombarding the subject stars from the left, with the x-axis aligned with the direction of the relative velocity of the field stars.

4.9.1 Hyperbolic orbits

We use now standard notations, with b the impact parameter of the field stars, ϕ the orbital plane angle and θ the deflection angle (see Chapter 1). The centre of mass velocity is equal to the field star velocity $\mathbf{V} \simeq \mathbf{v}_s$ and the relative velocity is equal to $\mathbf{v} \simeq \mathbf{v}_f - \mathbf{v}_s$. Note that the norm of the relative velocity will be conserved, but not its direction. The particle approaches from the left with $x = -\infty$, $y = b$, $v_x = v_\infty$ and $v_y = 0$. The trajectory is described by a Kepler orbit, with interaction potential $\phi(r) = -GM/r$. Using the solution for Kepler orbits that we have found earlier in this chapter, we can describe the trajectory with

$$u = C \cos(\theta - \theta_0) + \frac{GM}{L^2} \quad (4.415)$$

where the new variable u is defined by $u = 1/r$. $L = rv_\theta$ is the constant angular momentum and C is a constant yet to be determined. Using the conditions at $x \rightarrow -\infty$, we get easily the value of the angular momentum as $L = -bv_\infty$. Note that it is negative in our positively oriented polar coordinate system. The radial velocity of the star writes in polar coordinates

$$v_r = \dot{r} = Cr^2 \dot{\theta} \sin(\theta - \theta_0) = CL \sin(\theta - \theta_0) = -Cb v_\infty \sin(\theta - \theta_0) \quad (4.416)$$

Using again the conditions as $x \rightarrow -\infty$, which corresponds to $\theta = \pi$ and $r \rightarrow +\infty$, we find for the radial velocity

$$v_r \rightarrow -v_\infty = -Cbv_\infty \sin(\pi - \theta_0) \quad \text{or} \quad \sin \theta_0 = \frac{1}{Cb} \quad (4.417)$$

and for the inverse radius

$$u = \frac{1}{r} \rightarrow 0 = C \cos(\pi - \theta_0) + \frac{GM}{L^2} \quad \text{or} \quad \cos \theta_0 = \frac{GM}{Cb^2v_\infty^2} \quad (4.418)$$

Combining these two results gives us the value of the orbital phase as

$$\boxed{\tan \theta_0 = \frac{bv_\infty^2}{GM} = \frac{b}{b_{90}} \quad \text{with} \quad b_{90} = \frac{GM}{v_\infty^2}} \quad (4.419)$$

where we introduced the well-known impact parameter b_{90} corresponding to a 90 degree deflection angle. This phase corresponds to the apocenter of the orbit, in other words the point of closest approach. The deflection angle corresponds to the polar coordinate for which $v_r = +v_\infty$, which gives

$$\theta_{\text{defl}} = 2\theta_0 - \pi \quad (4.420)$$

We can compute the velocity of the escaping star after the collision as

$$v_x = v_{\parallel} = v_\infty \cos \theta_{\text{defl}} \quad \text{and} \quad v_y = v_{\perp} = v_\infty \sin \theta_{\text{defl}} \quad (4.421)$$

We are interested mostly in the parallel velocity kick, leading to a drag force due to collisions with field stars. The perpendicular velocity kick will average out with particles coming in with the opposite impact parameter. The parallel velocity kick is therefore

$$\Delta v_{\parallel} = v_\infty \cos \theta_{\text{defl}} - v_\infty = -v_\infty (1 - \cos \theta_{\text{defl}}) = -v_\infty (1 + \cos 2\theta_0) \quad (4.422)$$

Using some standard trigonometry, we get finally

$$\Delta v_{\parallel} = -\frac{2v_\infty}{1 + \tan^2 \theta_0} = -\frac{2v_\infty}{1 + \frac{b^2}{b_{90}^2}} \quad (4.423)$$

Note that this velocity kick is negative. It corresponds to the field star being decelerated by the interaction with the subject star. Momentum conservation implies that in return, the subject star will received a much smaller, but positive velocity kick in the x-direction, whose value is given by

$$\boxed{\Delta v_{s,\parallel} = \frac{m}{M} \frac{2v_\infty}{1 + \frac{b^2}{b_{90}^2}}} \quad (4.424)$$

4.9.2 Chandrashekar formula

We now want to compute the total momentum gained by the subject stars due to collisions with field stars with a given velocity \mathbf{v}_f and at the position of the subject star \mathbf{x}_s . Note that the number density of such stars is given by the distribution function as

$$\delta n_f = f(\mathbf{x}_s, \mathbf{v}_f) d^3v_f \quad (4.425)$$

Using the collision cylinder, we integrate over all possible impact parameters to get the net force applied to the subject star

$$\delta F_s = M \frac{dv_s}{dt} = \int_{b_{\min}}^{b_{\max}} M \Delta v_{s,\parallel} 2\pi b v \delta n_f db \quad (4.426)$$

where the relative velocity was defined above as $v = v_\infty = |\mathbf{v}_f - \mathbf{v}_s|$. We get

$$\delta F_s = \delta n_f 4\pi m v_\infty^2 \int_{b_{\min}}^{b_{\max}} \frac{b db}{1 + \frac{b^2}{b_{90}^2}} = \delta n_f 4\pi m v_\infty^2 b_{90}^2 \left[\ln \left(1 + \frac{b^2}{b_{90}^2} \right) \right]_{b_{\min}}^{b_{\max}} \quad (4.427)$$

We see that the lower bound of the integral can be taken as $b_{\min} = 0$ but the upper bound has to be finite. A conservative choice is here the size of the entire galaxy R . This leads us to define here again a Coulomb logarithm as

$$\ln \Lambda = \ln \left(1 + \frac{R^2}{b_{90}^2} \right) = \ln \left(1 + \frac{R^2 v_\infty^4}{G^2 M^2} \right) \quad (4.428)$$

so that the force contribution can be written as

$$\delta F_s = \delta n_f 4\pi m \frac{G^2 M^2}{v_\infty^2} \ln \Lambda \quad (4.429)$$

Note that this force contribution is positive, and aligned with the relative velocity vector $\mathbf{v}_f - \mathbf{v}_s$. We need to reformulate this force contribution in an absolute frame, not in the frame aligned with the direction of the relative velocity of the field stars. This can be done using now a vector representation as

$$\delta \mathbf{F}_s = \delta n_f 4\pi m G^2 M^2 \ln \Lambda \frac{\mathbf{v}_f - \mathbf{v}_s}{|\mathbf{v}_f - \mathbf{v}_s|^3} \quad (4.430)$$

In this absolute frame, we can now integrate over the field stars particle velocity distribution function. We get for the total force

$$\boxed{\mathbf{F}_s = 4\pi G^2 M^2 \ln \bar{\Lambda} \int_{\mathbb{R}^3} \frac{\mathbf{v}_f - \mathbf{v}_s}{|\mathbf{v}_f - \mathbf{v}_s|^3} f d^3 v_f} \quad (4.431)$$

Note that here again the particle mass disappeared, because we absorbed it in the distribution function definition which is normalised to ρ . We have also introduced the average Coulomb logarithm to account for its integration over velocity space. We now assume that the distribution function is isotropic, like the ergodic family $f(E)$ or the Maxwell-Boltzmann distribution. This means that the distribution function is spherically symmetric in velocity space. We now use an analogy with the gravitational force of a fluid with mass density $\rho(\mathbf{x})$ acting on a fixed point \mathbf{y} that writes

$$\mathbf{g}(\mathbf{y}) = \int_{\mathbb{R}^3} G \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \rho(\mathbf{x}) d^3 x \quad (4.432)$$

If the mass density is spherically symmetric $\rho(r)$, we can use the divergence theorem and compute directly the gravitational acceleration as

$$\mathbf{g}(\mathbf{y}) = -\frac{GM(< y)}{y^3} \mathbf{y} \quad (4.433)$$

where y is the radius with respect to the centre of symmetry of the system. We now exploit this analogy to re-write the integral of our isotropic distribution function as

$$\int_{\mathbb{R}^3} \frac{\mathbf{v}_f - \mathbf{v}_s}{|\mathbf{v}_f - \mathbf{v}_s|^3} f d^3 v_f = -\frac{\rho(< v_s)}{v_s^3} \mathbf{v}_s \quad (4.434)$$

The notation $\rho(< v_s)$ corresponds to the mass density of particles whose velocity is smaller than the subject star velocity. Only slow enough particles contribute to the drag force. We can now present the final expression of the dynamical friction force, known as Chandrashekar formula.

$$\mathbf{F}_s = -4\pi G^2 M^2 \ln \bar{\Lambda} \frac{\rho(< v_s)}{v_s^3} \mathbf{v}_s \quad (4.435)$$

In order to estimate the effect of dynamical friction on our massive object, we define the dynamical friction time scale as

$$\frac{1}{t_{\text{df}}} = \frac{1}{v_s} \frac{dv_s}{dt} \quad (4.436)$$

for a typical collisionless, nearly circular orbit. If the massive object finds itself on radius r_s within a galaxy of mass M_{gal} , we now that the typical orbital velocity is

$$v_s^2 \simeq \frac{GM_{\text{gal}}}{r_s} \quad (4.437)$$

Since the magnitude of the acceleration of the drag force is

$$\frac{dv_s}{dt} = 4\pi G^2 M \ln \bar{\Lambda} \frac{\rho}{v_s^2} \quad (4.438)$$

we get

$$\frac{1}{t_{\text{df}}} = 4\pi G^2 M \ln \bar{\Lambda} \frac{\rho}{v_s^3} \quad (4.439)$$

As always, we want to compare this time scale with the orbital time give by $t_{\text{orb}} = r_s/v_s$.

$$\frac{1}{t_{\text{df}}} = t_{\text{orb}} \frac{1}{r_s} 4\pi G^2 M \ln \bar{\Lambda} \frac{\rho}{v_s^2} = t_{\text{orb}} 4\pi G \rho \frac{M}{M_{\text{gal}}} \ln \bar{\Lambda} \quad (4.440)$$

where we injected the value of the circular velocity. Introducing the free-fall time of our collisionless stellar fluid, we finally get

$$\frac{1}{t_{\text{df}}} = \frac{t_{\text{orb}}}{t_{\text{ff}}^2} \frac{M}{M_{\text{gal}}} \ln \bar{\Lambda} \quad (4.441)$$

We still have to compute the average Coulomb logarithm. We can estimate it approximately by

$$\ln \bar{\Lambda} \simeq \ln \left(1 + \frac{R^2 \sigma^4}{G^2 M^2} \right) \simeq 2 \ln \frac{M_{\text{gal}}}{M} \quad (4.442)$$

where we used the fact that the typical stellar velocity dispersion satisfies $\sigma^2 \simeq GM_{\text{gal}}/R$. If the stellar mass is comparable to the dark matter mass, we find that the free-fall time is comparable to the orbital time. We finally get the result

$$t_{\text{df}} = \frac{t_{\text{orb}}}{2} \frac{M_{\text{gal}}/M}{\ln(M_{\text{gal}}/M)} = \frac{t_{\text{orb}}}{2} \frac{\mu}{\ln \mu} \quad (4.443)$$

It is quite striking that this formula resembles the relaxation time formula so much. This clearly suggests that dynamical friction is a relaxation process associated to collisions. It all depends on the mass ratio μ between the host galaxy and the massive object. For $\mu = 10$, we find $t_{\text{df}} \simeq 2t_{\text{orb}}$, while for $\mu = 100$, we find $t_{\text{df}} \simeq 10t_{\text{orb}}$. For the Milky Way, we have $M_{\text{gal}} \simeq 10^{10}$ solar masses. Only objects as massive as 10^8 solar masses will be able to deviate enough from a pure collisionless dynamics and see their trajectory modified by dynamical friction. This proves for example that the supermassive black hole at the centre of our galaxy (10^6 solar masses) has not been accreted from outside but most likely formed in situ. On the other hand, with an estimated stellar mass of 3×10^9 solar masses, the large Magellanic cloud will quickly spiral down towards the centre of the Milky Way.